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I. *Operational Factors in Orifice Flow.*

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THE volumetric rate of flow through an orifice is commonly expressed in the form

$$Q=C_d \, . \, a\sqrt{2gH},$$

where a is the area of the orifice, H the effective head, and C_d the coefficient of discharge. The usefulness of the quantity $a\sqrt{2gH}$ as a basis of comparison lies in the facts that it is easily calculated from simple observations, and that for most practical purposes the value of the coefficient C_d is tolerably constant.

This coefficient itself is merely a convenient name for the product of certain other factors or coefficients which can be more directly attributed to physical causes and are therefore of primary importance in scientific inquiry. These factors are: the coefficient of contraction C_c, which arises from the fact that the discharging jet forms a vena contracta; the coefficient of velocity C_v, due to the fact that this jet has a velocity less than that theoretically due to the change in pressure; and with compressible fluids the "attenuation factor" C_a, which allows for the fact that the volume of such a fluid when measured at its initial density is less than that of an incompressible liquid due to attenuation at reduced pressures.

The values of these contributory factors depend on a large

* Communicated by the Author.

number of conditions, which may generally be attributed either to the nature of the fluid in use or to the artificial configuration of the apparatus and the fashioning of the orifice itself. The effects of the physical properties of the fluid have been treated to some extent on a rational basis * and fairly completely by experiment, and the effects of constructional factors are also known empirically † for most cases of practical importance. There are, however, certain conditions of flow which may affect the registration of an orifice to an important extent, and which are not directly attributable to hydraulic or constructional conditions but rather to the imposed conditions of operation. It is proposed here to consider the effects of three phenomena which may be classified as operational factors: Turbulence, Cavitation, Pulsating flow.

TURBULENCE.

In problems concerning uniform flow in pipes it is generally accepted that there is no need to pay quantitative regard to the degree of turbulence in the flow, this being implied in the value of the Reynolds number. Some caution is required, however, in applying this principle to other types of flow, for in obtaining the experimental evidence on which it is at present based, precautions were taken to suppress turbulence as far as possible—in fact, the conditions were those of "minimum turbulence." In orifice flow it is found over a wide useful range that when similar precautions are taken the phenomena of flow can be explained in terms of the viscosity criterion which corresponds to the Reynolds number; but these conditions are not easy to reproduce in experiment, still less in practical applications, so that acceptance of the Reynolds number as the sole criterion in uniform pipe-flow does not establish for it a *prima facie* claim under the different conditions which prevail in orifice flow. This case must be decided on independent evidence.

The supposed similarity between the conditions of pipe and orifice flow seems to be based largely on work carried out independently in Australia ‡ and America §, as a result of which it has been maintained that the coefficient of discharge (for water) diminishes as the head increases until a

* Phil. Mag. Oct. 1926, p. 852.

† Muller, Z. V. D. I. lii. p. 285; Gaskell, Proc. Inst. C. E. cxcvii. p. 243; Hodgson, Proc. Inst. C. E. cciv. p. p. 131.

‡ Bilton, Proc. Vict. Inst. Eng. ix. p. 27.

§ Judd and King, Am. Soc. Adv. Sci. 1906.

head of the order of 18 inches is reached, and that above this "critical head" the conditions become essentially turbulent and the coefficient remains constant at a "normal" value which is dependent on the size of orifice and on that alone. Against this must be set the evidence of almost all other workers * in this field, who have obtained systematic changes in the coefficient at values of the Reynolds number far above that suggested as critical. The present author has himself been unable to obtain independent evidence of any critical change, but has found it possible (if no precautions are taken, easy) to obtain clear indications of turbulence with almost any head and orifice, and to a varying degree. The fact that all experimenters find it necessary to take steps to avoid turbulence is itself evidence that the Reynolds number is not always a sufficient criterion.

In considering the effects of turbulence, therefore, it is thought necessary to adopt a separate criterion, even if there are no means of evaluating it, and the simplest form appears to be the ratio $\frac{l}{\epsilon}$, where l stands for the linear dimensions of the apparatus and ϵ is representative of the dimensions of the eddy system †. It is to be remarked that in general the degree of turbulence will increase as the statistical size of the eddies decreases (as, indeed, appears to be the case in pipe flow), and therefore as the value of the criterion increases.

Supposing in the first instance that turbulence alone is operative, we may express the total loss of head in the contracting jet by means of the equation

$$\mathrm{H}_x = \frac{\mathrm{V}^2}{2g} \cdot a\tau,$$

where $\tau = f\left(\frac{l}{\epsilon}\right)$, and may be defined as the "intensity of turbulence."

Under these conditions the coefficient of velocity is given by $\mathrm{C}_v = (1 + a\tau)^{-\frac{1}{2}}$, and for practical purposes $= 1 - \frac{a\tau}{2}$ nearly, showing that it falls short of unity by an amount simply proportional to the "intensity of turbulence."

* *Cf.* Hamilton Smith, 'Hydraulics,' ch. iii.; Dempster Smith and Walker, Proc. I. Mech. E. 1923, p. 23.

† It is what may be termed accidental as opposed to systematic turbulence which is here under discussion. Conditions which give rise to systematic vertical motion are clearly not amenable to such simple treatment, though they are generally found to cause increased discharge from a sharp-edged orifice.

In order to examine rationally the consequence of changes in the coefficient of contraction, we shall assume that the loss of head incurred by the fluid in its motion up to that surface, normal to the stream-lines, which contains the bounding curve of the orifice, may be written

$$h_x = \frac{v^2}{2g} \cdot b\tau,$$

where v is "representative" of the velocity at this surface. Following, then, the line of argument advanced in an earlier paper for the case of viscous resistance *, we are led to an analogous expression for the coefficient of contraction:

$$C_c = C_0 + b\tau(1-C_0)(2-C_0),$$

and so for the coefficient of discharge:

$$C_d = C_0 + b\tau(1-C_0)(2-C_0) - C_0\frac{a\tau}{2}.$$

It appears, therefore, that turbulence will tend to increase the area of the jet at the same time that it reduces the velocity of discharge. The resultant effect on the coefficient of discharge depends on the comparative values of a and b. If for descriptive purposes we take $a=b$, then the coefficient will vary according to the relation

$$C_d = C_0 + a\tau(2 - \tfrac{7}{2}C_0 + C_0^2),$$

and the changes so produced for various values of C_0 are plotted in fig. 1. From this it seems likely that for Borda and sharp-edged orifices turbulence will cause an increase in the discharge under normal hydraulic conditions, while for rounded orifices it will definitely give rise to a diminution.

The special case of the Borda mouthpiece admits of no doubt when considered in view of the forces causing discharge; turbulence, or indeed any form of fluid friction, must give rise to a progressive increase in the coefficient of discharge so long as the mouthpiece continues to "run free."

For each form of orifice we are led to the conclusion that turbulence acting alone will effect a change in the coefficient which is proportional to the intensity of turbulence, and the coefficient will be constant if and only if this intensity remains constant.

Supposing, now, that it is possible for turbulent and viscous resistance to operate together, it is easy to apply the argument advanced above and to show that the resulting change in the

* Phil. Mag. Oct. 1926, p. 867.

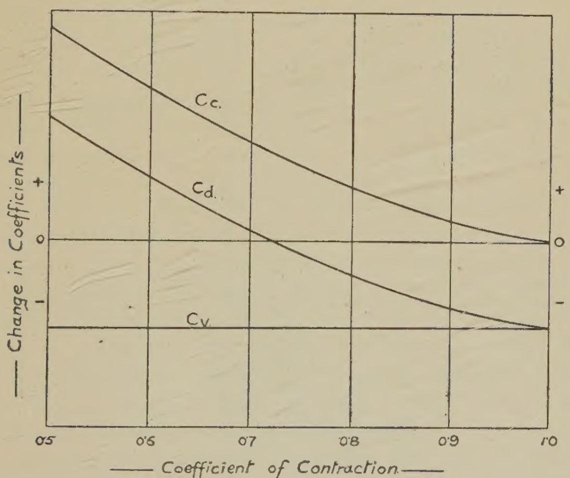
coefficient is the same as if the separate effects were superposed, so that

$$C_d = C_0 + c_1\eta + c_2\tau.$$

If it be further supposed that the two types of resistance are independent, then turbulence will still give rise to a change in the coefficient, as modified by viscous effects, which is dependent simply on its "intensity" τ.

The joint operation of turbulence and viscosity is a matter for appeal to experiment. In fig. 2 are shown results obtained by timing the fall in level of water in a tank 3 feet in diameter as it discharged through a sharp-edged orifice $\frac{1}{2}$ inch in diameter mounted in the base. In one case the water was

Fig. 1.

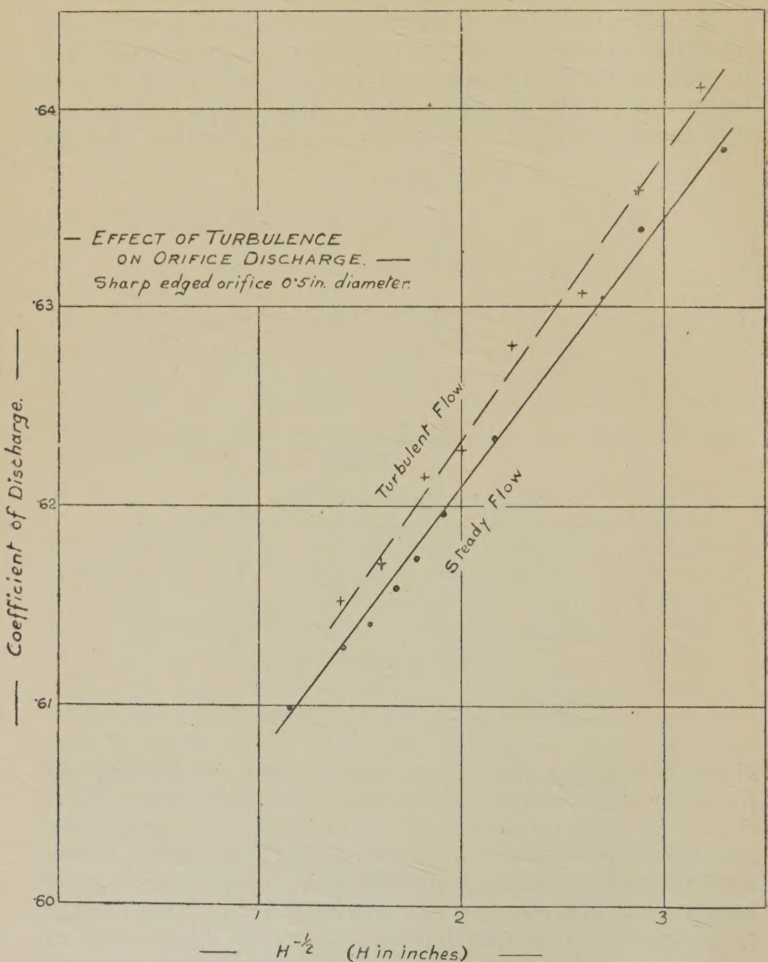


EFFECT OF TURBULENCE ON COEFFICIENTS.

allowed to become still before discharge was commenced; in the other, turbulence was maintained artificially throughout discharge by unsystematic stirring as uniform in degree as possible. The results show that with the sharp-edged orifice used turbulence causes a definite increase in the discharge, but does not give rise to a constant coefficient. The changes in the coefficient are, in fact, systematically identical with those obtained under quiescent conditions, showing an increase which, although variable, has a mean value of about 0·5 per cent. When the same tank was emptied under naturally turbulent conditions (*i. e.* without waiting for the water to still before commencing discharge), the results

were similar, but showed a smaller increase of about 0·3 per cent. over the steady values. Determinations made at constant heads of 12 and 6·5 inches under artificially turbulent conditions showed on the average increases of $0{\cdot}5_5$ and 0·6 per cent. respectively. Tests with an orifice

Fig. 2.



1·0 inch diameter under a falling head gave an average rise of 0·4 per cent. in the coefficient with artificial turbulence. Under conditions of initial natural turbulence, a vortex formed at heads between 40 and 50 inches and deprived the results of any comparative value.

These experiments confirm the view that turbulence in orifice flow requires a separate criterion; they show definitely that turbulence and viscosity may be operative at the same time, and afford evidence that under the conditions of hydraulic discharge their effects may be superposed. Since the turbulence artificially maintained in these tests was considerably greater than that likely to be encountered in practice, the results give also some idea of the maximum effect likely to be introduced by turbulence in orifices of the sizes used.

The importance of turbulence will be determined by the ratio $\frac{l}{\epsilon}$. As a general rule, it is to be expected therefore that for a given eddy system the effects of turbulence will increase with the dimensions of the orifice, though the somewhat rough methods of the experiments just described do not give any indication of this. In the same way the statistical dimensions of the eddies will become smaller as the velocity and head increase; so that these conditions also are likely to make turbulence more important. In practice, moreover, turbulence is by nature inconstant; and since fluctuations in its intensity will be reproduced in the discharge, it is to be expected that conditions which predispose to turbulence will also give rise to variable registration. Furthermore, the factors which increase the importance of turbulence are those also which reduce the effect of the viscosity criterion η. Hence it is to be expected that with greater heads, larger orifices, and lower viscosities the results of experiments will be less consistent, and that under the conditions of approach normally met the coefficient of discharge may apparently cease to vary systematically. These systematic variations, due to the effects of viscous resistance and other causes, should, however, become evident when precautions are taken to eliminate the effects of turbulence.

The apparent constancy of the coefficient of discharge in the experimental work on which the "critical head" theory is based, may be fairly attributed to turbulent conditions in approach and to the accidental errors of experiment which might easily mask the small variations in the coefficient with large orifices and great heads. In point of fact, the expected change in the coefficient for a 2-inch orifice due to viscosity is only ·01 between heads of 2 and 100 feet, which is of the same order as the irregular variations in the results of the experiments referred to. Moreover, in some

of these experiments * it is stated that turbulence "was very noticeable at high heads"; while Dempster Smith †, who worked over a similar range, was able to maintain a clear glass-like surface and also a continuously falling coefficient.

An interesting fact which seems further to justify the similar treatment given above to viscous and to turbulent resistance was made evident in experiments by the author with a partially rounded orifice. This orifice was devised, by trial and error, of such a form that decrements in the coefficient of velocity due to viscous resistance were almost exactly balanced by corresponding increments in the coefficient of contraction, so that a virtually constant coefficient of discharge was maintained with water over a considerable range of heads and temperatures. It was found that the discharge from this orifice was also practically unaffected by turbulence in approach. When tested under a falling head between levels of 55 and 5 inches, the times recorded were: quiescent at 14° C., 1149·9 secs.; quiescent at 63° C., 1151·4 secs.; turbulent at 14° C., 1150·2 secs.

Cavitation.

The registration of an orifice or similar form of meter may be affected to a considerable extent by the existence of a low pressure at the vena contracta. If the velocity at this point is high, the pressure may become correspondingly so low as to give rise to cavitation.

As the pressure of a liquid free from dissolved gases is reduced, no effect of hydraulic interest occurs until it reaches the vapour-pressure corresponding to the temperature of the liquid. At this point evaporation takes place and the constitution of the mixture passing the vena contracta is not easily determinable. Under these circumstances, of course, meters of the orifice type cease to have any value. In practice it will usually be easy to avoid such a state of affairs, though it may arise, for example, if a venturimeter is employed with hot water under a low static head.

A disturbing condition of greater practical interest arises from the presence of dissolved gases in the liquid passing through the meter. As the pressure of such a solution is reduced, the solubility of the gases is diminished according to Henry's law; and if the pressure falls below that at which

* Judd and King, *loc. cit.*
† Proc. I. Mech. E. 1923.

the solution becomes saturated, the excess gases will be set free and cavitation will occur. This effect will be further complicated by the partial evaporation of the liquid itself in the gas so liberated. This condition, though it may occur at a considerably higher pressure than that at which mass evaporization takes place, is not, as will appear, likely to become important in the case of a sharp-edged orifice, which owing to inferior pressure recovery is seldom employed in a pipe under large differential heads; but it is quite easy to produce in certain types of venturimeter, and its effect can then be estimated with some measure of confidence.

Suppose in the first instance that the water passing through a meter was saturated with (3 per cent. by volume of) air at atmospheric pressure. At an absolute pressure p inches of mercury the proportion of air remaining in solution will be $\frac{p-v}{30}$, where v is the vapour-pressure at the temperature which obtains. Hence the volume of air set free per unit volume of water is $\frac{3}{100}\left(1-\frac{p-v}{30}\right)\frac{30}{p-v}$, measured at the existing throat-pressure. Hence the fraction k of the throat area actually occupied by water will be given by

$$\frac{1}{k}=1+\frac{9}{10}\left(\frac{1}{p-v}-\frac{1}{30}\right),$$

and the coefficient of discharge C becomes modified to C′ where

$$\frac{C'}{C}=\sqrt{\frac{m^2-1}{\frac{m^2}{k^2}-1}},$$

m being the ratio of pipe to throat areas.

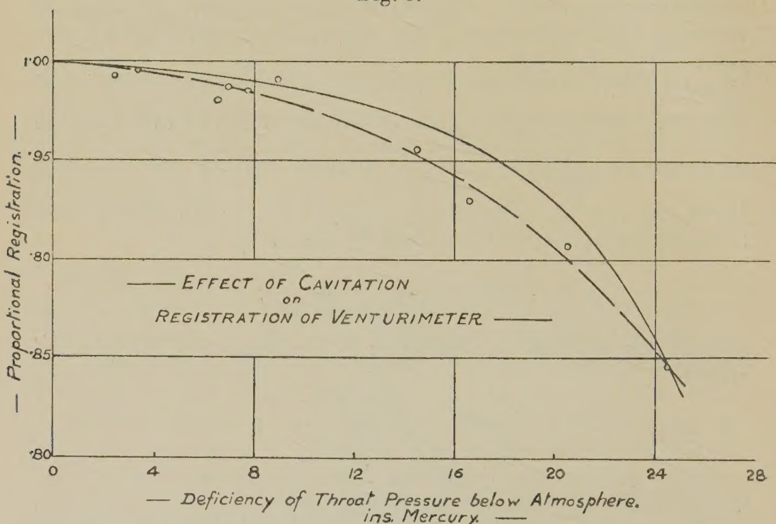
The changes represented by this expression, when applied to the case of a venturimeter for which $m=9$, delivering water at a temperature of about 23° C., are indicated by the full line in fig. 3. Results deduced from a series of tests under these conditions are shown by points and the dotted line in the same figure.

In these tests a venturimeter was employed mounted in a six-inch pipe-line at the end of a straight length of upwards of forty diameters. At the entry to this length was a branch containing a six-inch valve which allowed more or less of the water to be bye-passed to waste. About the same distance beyond the venturi tube, there was mounted in the pipe a glass section, and just beyond this a delivery valve through which the water passed to a triangular notch meter. Water

was supplied from a turbine pump. It will be seen that by manipulating the delivery and bye-pass valves the rate of discharge and the pressure at the throat at the venturimeter may be varied independently. In the tests quoted it was clearly the best course to maintain a tolerably constant rate of flow at various throat-pressures in order to eliminate variations due to other causes in the coefficient either of the venturimeter or notch.

The differential pressure tubes were led from the lower parts of the annular rings at the entry and throat of the venturi tube, and a continual fall was maintained to the gauge itself. The pressure lead from the venturi throat to the

Fig. 3.



absolute pressure gauge was similarly taken from the lower part of the throat, and correction was made for differences of level. Freedom from air-locks was tested before and after each experiment.

As was to be expected, particularly at the higher vacua, the venturimeter readings were unsteady, and there was a certain amount of oscillation in the gauge-tubes. Nevertheless, the results are in general agreement with the theoretical values, and show that a more or less systematic diminution in the coefficient occurs at low throat-pressures, and that this may under certain circumstances be very considerable. During the tests at the lower pressures cavitation was distinctly audible at the throat of the meter, and bubbles

of air were visible in the water at the glass section of the pipe some forty diameters beyond.

In the equation given above, the effect of any value of k is greater for small than for larger values of m, but in practice low pressures are more likely to occur with high values of m. For example, with a pipe velocity of 5 feet per second the necessary static head in the pipe just above the venturimeter to ensure that the pressure at the throat shall not fall below atmospheric is 31 feet with a pipe to throat ratio of 9 and less than 6 feet with a ratio of 4. Cavitation is therefore only likely to arise to any considerable extent in practice when a meter of rather large pipe to throat ratio is installed near the open end of a pipe-line, and since modern design tends to favour small values of this ratio, cavitation is becoming less and less significant. Moreover, since the numerical case cited above refers to complete saturation of the water at atmospheric pressure, it gives a limiting value beyond which the correction for water is never likely to rise, and shows that this correction is quite small unless the throat-pressure is much below atmospheric.

There appear to be no previously published results against which these conclusions can be checked. The corrections proposed are scarcely effective, for example, under the conditions quoted by Gibson *, which refer to throat-pressures in no case more than 18 inches of water below atmospheric. The discussion will, of course, only apply provided air-locks are eliminated in the gauge and its connexions, and non-observance of this condition has probably been responsible for more inaccurate registration in practice than has cavitation at the throat itself. The presence of air-locks is, of course, fatal to consistent metering.

Pulsating Flow.

For the measurement of a flow subject to fluctuations of short period, the orifice operates under a disadvantage inherent in any form of meter whose registration depends on a differential pressure not directly proportional to the rate of flow. This disadvantage arises not from irregularities at the orifice itself or changes in the actual coefficient of discharge, but from the means which have to be employed to measure the rapidly changing pressure difference. In order to obtain satisfactory readings it is necessary to damp out the oscillation of the fluid column in the gauge and its connexions ; and

* Proc. Inst. C. E. cxix. p. 391.

unless special and rather elaborate methods are employed, the mean value of the pressure so determined is representative not of the average $(\bar{q})$, but of the "root mean square" (q) rate of flow through the orifice.

In order, therefore, to compute the true mean rate of flow, it is necessary to multiply by the factor $\frac{\bar{q}}{q}$ the rate of flow deduced from the mean value of the pressure difference. The value of this factor will, of course, depend upon the form of the pulsations and upon the frequency of their recurrence, and can only be calculated when these are known.

a. *Liquid Flow.*

In the case of liquid flow entirely confined by solid boundaries, the value of this factor can usually be determined with reasonable accuracy from the known conditions at the source of pulsation, since these will as a rule be exactly reproduced at the orifice.

For a flow curve consisting of sinusoidal arcs, such as might be produced by a two-throw single-acting pump without an air vessel, the value of the factor is easily found to be $\frac{\sqrt{8}}{\pi} = 0{\cdot}90$ approx., and is therefore of importance. For a three-throw single-acting pump, on the other hand, the factor differs from unity by one part in 1200 only, so that no correction is necessary.

The single-ram pump without air vessel gives a discharge curve of the same individual form as the two-throw, but the pulses are intermittent. It is easily shown that the factor for any such intermittent flow may be obtained from the "form factor" (the factor obtained from the wave-form itself) by the use of an "intermittency factor" $\sqrt{k}$, where k is the ratio of the duration of the pulse to the period of the complete cycle.

For if Q is the rate of flow at any instant, and t_0 the period of the complete cycle, we may write

$$\frac{\bar{q}}{q} = \frac{\frac{1}{t_0}\int_0^{t_0} Q\,dt}{\left[\frac{1}{t_0}\int_0^{t_0} Q^2\,dt\right]^{\frac{1}{2}}};$$

and since each of these integrals is unchanged in value except during the portion kt_0 of the period, this expression

may be written

$$\frac{\frac{1}{kt_0}\int^{kt_0} Q\,dt}{\left[\frac{1}{kt_0}\int_0^{kt_0} Q^2\,dt\right]^{\frac{1}{2}}}\sqrt{k} = f\cdot\sqrt{k},$$

where f is the "form factor."

Hence in the simple case of a single-ram pump the ratio

$$\frac{\bar{q}}{q} = \frac{\sqrt{8}}{\pi}\cdot\sqrt{\tfrac{1}{2}} = 0{\cdot}636.$$

In practice, of course, it is seldom necessary to measure the discharge of such a pump without an air vessel, and an efficient air vessel will greatly reduce the importance of the correction. For example, a continuous sinusoidal fluctuation

$$Q = q_0 + q_1 \sin \omega t$$

gives a factor *

$$\left(1 + \frac{1}{2}\cdot\frac{q_1{}^2}{q_0{}^2}\right)^{\frac{1}{2}},$$

showing that the correction is less than 1 per cent. so long as the amplitude of these fluctuations does not exceed 40 per cent. of the mean rate of flow.

b. *Gaseous Flow.*

The effect of fluctuations on the registration of an orifice passing a gas or vapour is not amenable to such simple treatment. The instantaneous rate of flow at different points of the system is dependent not only on the changes at the source of pulsation, but also on the elastic properties of the fluid and on the damping effect of capacity and friction between the points of measurement and imposed pulsation.

Theoretical treatment of the effect of capacity for a given impressed wave-form leads to differential equations which can only be solved by a method of trial and error, all the more troublesome in that each trial involves a step-by-step integration †. In this treatment the impressed wave-form is usually a matter of some uncertainty, and such effects as friction and mass acceleration cannot well be taken into account. The results themselves are therefore to be regarded as essentially approximate, and can more usefully be employed to determine the precautions necessary to keep the correction within defined limits rather than to compute this correction

* Gibson, P. R. S. Edin. xxxiii. p. 103.

† Hodgson, Inst. C. E. Sel. Pap. xxxi. p. 13.

when large. For this purpose a fairly simple method of treatment may be applied.

1. *Instantaneous Pulses.*

For a given mean rate of flow $\bar{q}$ the effect of pulsation is clearly greatest when the whole quantity passing during a complete cycle is discharged into or out of the system instantaneously, and precautions which enable satisfactory registration to be made under these conditions may be regarded as safe in all cases.

In the discussion which follows we shall suppose that the gas after passing the orifice enters a "receiver" of volume V from which it is drawn in pulses, by an engine for example.

Assume that the pressure p_0 upstream of an orifice remains sensibly constant, and that the density of the gas is then ρ_0. If the pressure and density in the receiver are p and ρ at any moment, the mass rate of flow through the orifice is given by

$$\frac{dm}{dt} = \rho_0 a_0 \sqrt{\frac{2(p_0-p)}{\rho_0}},$$

where a_0 is the effective area of the vena contracta. The effect of compressibility, if not negligible, may under all conditions of practical metering be allowed for sufficiently by a correction to a_0 corresponding to the mean pressure difference Δp.

The throttling effect of the orifice will for a permanent gas* give nearly isothermal conditions in any adequate receiver, and it follows that

$$\frac{\rho_0}{\rho} = \frac{p_0}{p},$$

so that the pressure in the receiver varies according to the relation

$$\frac{dp}{dt} = \frac{p_0 a_0}{\mathrm{V}} \sqrt{\frac{2(p_0-p)}{\rho_0}}.$$

Writing the volumetric rate of flow as q, we find

$$\frac{dq}{dt} = -\frac{p_0 a_0^2}{\rho_0 \mathrm{V}},$$

which gives a linear relation between the rate of flow and time:

$$q = q_1(1-rt),$$

where q_1 is the initial value of q—*i.e.*, the rate of flow

* This argument cannot, of course, be maintained for steam, but the results are not likely to be greatly at variance.

immediately after the instantaneous discharge from the receiver—and $r = \dfrac{p_0 a_0^2}{\rho_0 V q_1}$.

Provided the period of fluctuation t_0 is not so great that the pressures p and p_0 equalize, the value of the ratio

$$\frac{\bar{q}}{q} = \frac{\left(1 - \frac{r}{2} t_0\right)}{\left(1 - r t_0 + \frac{r^2}{3} t_0^2\right)^{\frac{1}{2}}},$$

which is within 2 per cent. of unity so long as $r t_0 < \frac{1}{2}$, and within 1 per cent. when $r t_0 < \frac{2}{5}$.

When the conditions are such that $r t_0 = \frac{1}{2}$, the final rate of flow is one-half the initial, so that $\bar{q} = \frac{3}{4} q_1$, and the volume of the receiver necessary to give these conditions is

$$V = \frac{3}{2} \cdot \frac{p_0 a_0^2 t_0}{\rho_0 \bar{q}} = \frac{3}{4} Q \cdot \frac{p_0}{\Delta p},$$

where

Q is the total discharge per cycle,

Δp is for practical purposes the mean presssure difference across the orifice.

The capacity so determined may be regarded as adequate to reduce pulsation effects to 2 per cent. under the most unfavourable conditions of supply, and if the capacity is as great as $Q \dfrac{p_0}{\Delta p}$ the discrepancy will be less than 1 per cent. On the other hand, if the capacity is much less than these values the pulsation effect will be considerable. A capacity of $\frac{3}{8} Q \dfrac{p_0}{\Delta p}$, for instance, gives an "error" of over 13 per cent. If the capacity is still less than this, there will be a period during each cycle over which there will be no flow from the orifice, and the "intermittency factor" $\sqrt{k}$ will then operate. It is more satisfactory, however, to prevent than to remedy such conditions in practice.

2. *Intermittent Pulses; constant rate.*

In order to obtain corresponding results for flow in which the pulses are not instantaneous but occupy a measurable part of the cycle, we shall consider next a flow which at the source of pulsation takes the form indicated by $A_1OB_1C_1C_2$ in fig. 4, having a constant rate C during the fraction k of the period t_0 and ceasing over the remainder of the cycle.

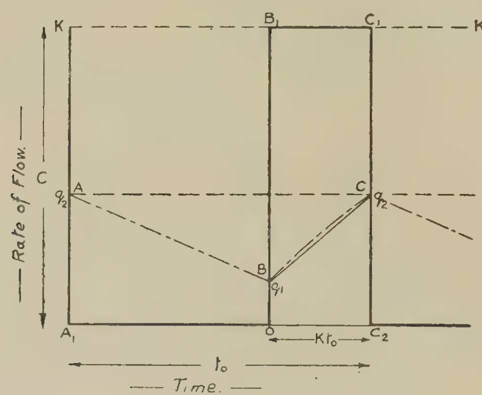
During these intervals the rate of flow at the orifice changes according to the linear law already considered. While discharge from the receiver is taking place at the rate C, the equation of flow through the orifice is

$$q = a_0\sqrt{\frac{2(p_0-p)}{\rho}},$$

where p satisfies the equation

$$\frac{dp}{dt} = \frac{p_0}{V}(q-C).$$

Fig. 4.



— PULSATING FLOW —

The solution of these equations is

$$C \log\frac{C-q}{C-q_1} + (q-q_1) + \frac{p_0 a_0^2}{\rho_0 V}t = 0,$$

showing that q approaches logarithmically the value C from its initial value q_1.

The variations in flow through the orifice during a cycle will therefore follow the line ABC in fig. 4, in which AB is a straight line and BC a logarithmic curve asymptotic to KK and of such a form that the areas A_1ABO and BB_1C_1C are equal. By direct integration and manipulation it can be shown that the ratio

$$\left(\frac{\bar{q}}{q}\right)^2 = k\frac{Q}{Q_0},$$

where

Q_0 is the volume of gas passing the orifice while delivery is taking place,

Q is the total discharge per cycle.

In cases where the effect of pulsation is small, which are in fact those under consideration, the arc BC will be but slightly curved, and either of two approximations may be employed to determine the necessary receiver capacity.

a. If we assume that the arc BC is parabolic, we obtain for the area between the arc and chord BC an expression which finally reduces to

$$a = \frac{1}{12} \cdot \frac{1-k}{\mathrm{C}} x_0^3,$$

where

$$x_0 = \frac{p_0 a_0^2}{\rho_0 \mathrm{V}} t_0.$$

Hence

$$\left(\frac{\bar{q}}{q}\right)^2 = \frac{k \times \text{total area under ABC}}{\text{OBCC}_2} = \frac{t_0(q_1+q_2)+2a}{t_0(q_1+q_2)+2\frac{a}{k}};$$

and using the approximate expression for a, we find

$$\left(\frac{\bar{q}}{q}\right)^2 = 1 - \frac{1}{12}\left(\frac{p_0 a_0^2 t_0^2}{\rho_0 \mathrm{VQ}}\right)^2 (1-k)^2$$

$$= 1 - \frac{1}{48}\left(\frac{\mathrm{Q}}{\mathrm{V}} \cdot \frac{p_0}{\Delta p}\right)^2 (1-k)^2 \text{ nearly,}$$

and the ratio $\frac{\bar{q}}{q}$ is within 2 per cent. of unity so long as $\mathrm{V} \geq \frac{3}{4}\mathrm{Q}\frac{p_0}{\Delta p}(1-k)$.

b. If we use the fact that the ratio $\frac{\bar{q}}{q}$ is not greatly different from that obtained when the rate of flow through the orifice follows the lines AB, BC, then by a simple integration we find

$$\frac{\bar{q}}{q} = \frac{1 - \frac{r_1 t_0}{2}}{\sqrt{1 - r_1 t_0 + \frac{r_1^2 t_0^2}{3}}}$$

where

$$r_1 = \frac{p_0 a_0^2}{\rho_0 \mathrm{V} q_2}(1-k),$$

a result similar to that obtained for instantaneous pulses but with $r(1-k)$ written in place of r. The correction for fluctuation becomes less than 2 per cent. when $rt_0(1-k) < \frac{1}{2}$, and this condition is satisfied when $\mathrm{V} \geq \frac{3}{4}\mathrm{Q}\frac{p_0}{\Delta p}(1-k)$, as above.

3. *Intermittent Pulses; variable rate.*

For a periodic demand on the receiver which is both intermittent and variable, the solution, as mentioned above, is troublesome, and at the best only leads to approximate results; particularly so when the exact wave-form is a matter of uncertainty. It is not difficult, however, to obtain a simple expression for the receiver capacity which will assure reasonably accurate registration.

Suppose the demand per cycle is Q and the maximum rate of flow q_m at the source of pulsation. Then the conditions with any normal wave-form are more favourable than those which would obtain if the whole demand Q were absorbed at a uniform rate q_m, with idle intervals. The receiver capacity necessary to reduce the correction to a given value under these hypothetical conditions would therefore be ample under the actual conditions of operation; and since the value of this capacity can be easily estimated, it would appear to afford a useful criterion in metering practice. This capacity is given by

$$V = \tfrac{3}{4} Q \frac{p_0}{\Delta p}\left(1 - \frac{\bar{q}}{q_m}\right)$$

to give a correction less than 2 per cent., and similarly by

$$V = Q \frac{p_0}{\Delta p}\left(1 - \frac{q}{q_m}\right)$$

to give a correction less than 1 per cent.

It will be found that these values are consistent with solutions obtained for special cases by Hodgson *, using step-by-step integration.

II. *On the Stability of the Solutions of Mathieu's Equation.* *By* Balth. van der Pol, *D.Sc., and* Dr. M. J. O. Strutt, *E.I.*†

THE movement of a particle in a field of force, the force being direct or inversely proportional to the elongation, is determined by the equation:

$$\frac{d^2x}{d\tau^2} + \omega_0^2 \, . \, x = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the case where the force is inversely proportional to the elongation, we have

$$\omega_0^2 > 0,$$

* *Loc. cit.* p. 14.

† Communicated by the Authors.

and the movement is of purely periodic type. On the other hand, if the force is directly proportional to the elongation, we have

$$\omega_0^2 < 0,$$

and the movement may be called unstable, as in general the elongation increases indefinitely with time.

If the force is a periodic function of the time with angular frequency p, the equation is:

$$\frac{d^2x}{d\tau^2} + (\omega_0^2 + \alpha_0^2 \,.\, \cos p\tau) \,.\, x = 0. \quad . \quad . \quad . \quad (2)$$

The three parameters occurring in this equation may simply be reduced to two parameters:

$$\frac{d^2x}{dt^2} + (\omega^2 + \alpha^2 \,.\, \cos t) \,.\, x = 0, \quad . \quad (3)$$

with

$$t = p \,.\, \tau,$$

$$\omega \,.\, p = \omega_0,$$

$$\alpha \,.\, p = \alpha_0.$$

The equation with a frictional term, assuming friction to be proportional to velocity, may be reduced to equation (3) by a well-known simple transformation.

The equation (3) is of the type generally known as *Mathieu's* differential [(1)*] equation.

In astronomy this equation has also been called after Lindstedt [(2)] and after Gyldèn [(2)].

If, instead of the term

$$\alpha^2 \,.\, \cos t,$$

we have a general Fourier series

$$\begin{aligned} F(t) = \omega^2 &+ A_1 \cos t + A_2 \cos 2t + A_3 \,.\, \cos 3t + \ldots \\ &+ B_1 \sin t + B_2 \sin 2t + B_3 \sin 3t + \ldots, \end{aligned}$$

so that equation (3) becomes of the form

$$\frac{d^2x}{dt^2} + x \,.\, F(t) = 0, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $F(t)$ is a function of t with period 2π, the equation (4) is called *Hill's* equation.

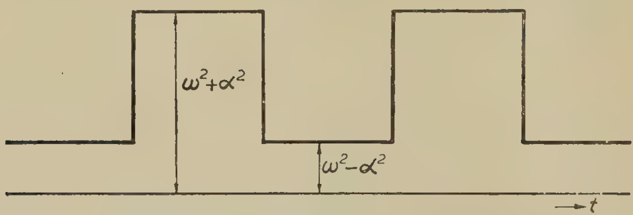
* These numbers refer to the bibliography at the end of the paper.

As a special form of equation (4), we have

$$\frac{d^2x}{dt^2}+x.\left[\omega^2+\frac{4}{\pi}\alpha^2(\cos t-\tfrac{1}{3}\cos 3t+\tfrac{1}{5}\cos 5t-\ldots)\right]=0, \quad (5)$$

so that the "*ripple*" in the force, acting on the particle under consideration, has the form of fig. 1. The equation (4) and its special forms (3) and (5) occur in several

Fig. 1.



problems, of which we only mention the most important ones, viz.:—

(1) The movement of a pendulum, the support of which is moved up and down with a frequency $p/2\pi$ [(3)];
(2) the oscillations of locomotive mechanisms [(4)];
(3) the movement of a stretched string, the tension of which is varied periodically [(5)];
(4) the propagation of waves in stratified media [(6)];
(5) the propagation of electromagnetic disturbances in cables of periodic structure (pupin-coils);
(6) the theory of modulation in wireless [(7)];
(7) the theory of superregeneration [(8)];
(8) the theory of astronomic perturbations [(9)];
(9) the production of electric currents by periodically varying the properties of circuits [(10)];
(10) the oscillations of membranes with elliptic boundary [(11)];
(11) the eddy currents in elliptic cylinders [(12)];
(12) the diffraction of light by elliptic cylinders [(13)];
(13) the oscillations of strings with periodic mass-distribution.

By Floquet's theorem [(1)], the general solution of equation (4) has the form

$$A\,.\,e^{\mu t}\,.\,\Phi(t)+B\,.\,e^{-\mu t}\,.\,\Psi(t),$$

where A and B are constants of integration,
Φ and Ψ are periodic functions of t with period 2π,
μ a coefficient independent of t, generally called "exposant caractéristique."

After this theorem, if

$$F(t)$$

is an arbitrary solution of equation (4), we have

$$F(t+2\pi) = \sigma \,.\, F(t).$$

Now three cases may arise:

$$(a) \quad |\sigma| > 1,$$
$$(b) \quad |\sigma| < 1,$$
$$(c) \quad |\sigma| = 1.$$

Case (*a*) represents a movement of unstable type; case (*b*) is of stable type; and case (*c*) is of unstable type.

It is to be noted that we call a movement stable if both solutions of equation (4) are of stable type, and unstable, if at least one of the solutions is of unstable type. From this definition it is already clear that there are more unstable possibilities than stable ones.

The first purpose of this paper is to fix the boundaries in the (a^2, ω^2)-plane between the stable modes of movement and the unstable ones. Moreover, the instability itself will be dealt with quantitatively.

In the literature the rules given for the stability of movement represented by equation (3), are partly contradictory [2] [14] [15]. This point makes it desirable to state clearly the conditions under which the solutions of (3) are stable.

Analytically equation (5) may be attacked in the simplest way, so that we shall first consider this case. A solution of equation (5) was already given by E. Meissner [4] for the case:

$$\omega^2 > \alpha^2 > 0.$$

The coefficient σ can be found in the following way:—

Let f and ϕ be two independent solutions of equation (4), and let F also be a solution, then

$$F(t) = \alpha_1 \,.\, f(t) + \alpha_2 \,.\, \phi(t).$$

Now we have

$$F(t+2\pi) = \sigma \,.\, F(t) = \alpha_1 \,.\, \;(t+2\pi) + \alpha_2 \,.\, \phi(t+2\pi),$$

where

$$f(t+2\pi) = a \,.\, f(t) + b \,.\, \phi(t),$$

$$\phi(t+2\pi) = c \,.\, f(t) + d \,.\, \phi(t).$$

Then, if we assume:

$$\phi(0) = 0, \quad \phi'(0) = 1,$$
$$f(0) = 1, \quad f'(0) = 0,$$

we have:

$$a = f(2\pi), \quad c = \phi(2\pi),$$
$$b = f'(2\pi), \quad d = \phi'(2\pi),$$

and

$$\mathrm{F}(t+2\pi) = \alpha_1(af+b\phi)+\alpha_2(cf+d\phi).$$

Hence

$$\alpha_1 a + \alpha_2 c = \sigma \,.\, \alpha_1,$$
$$\alpha_1 b + \alpha_2 d = \sigma \,.\, \alpha_2,$$
$$\frac{\alpha_1}{\alpha_2} = -\frac{c}{a-\sigma} = -\frac{d-\sigma}{b}. \quad . \quad . \quad . \quad . \quad (6)$$

From the differential equation it follows that:

$$f \,.\, \phi' - \phi \,.\, f' = \text{constant},$$

and in our case

$$f \,.\, \phi' - \phi \,.\, f' = 1 = a \,.\, d - b \,.\, c. \quad . \quad . \quad . \quad (7)$$

From equations (6) and (7) we conclude:

$$\sigma^2 - \sigma(a+d) + 1 = 0,$$

or

$$\sigma = \frac{a+d}{2} \pm \sqrt{\left(\frac{a+d}{2}\right)^2 - 1} \quad \text{and} \quad \frac{a+d}{2} = \cos 2\pi\mu. \quad (7')$$

Hence

$$\sigma = e^{\pm 2\pi j\mu}, \quad j = \sqrt{-1}.$$

The three different possibilities of movement, mentioned above, are now expressed by:

(*a*) $|\cos 2\pi\mu| > 1$: unstable,

(*b*) $|\cos 2\pi\mu| < 1$: stable (indifferent),

(*c*) $|\cos 2\pi\mu| = 1$: unstable.

Case (*c*) constitutes the limiting case of (*a*) and (*b*).

Now, in the case of a rectangular "*ripple*," with which we will first deal, we may consider two intervals, corresponding to fig. 1:

(I.) from $t = 0$ till $t = \pi$: $\dfrac{d^2x}{dt^2} + (\omega^2 + \alpha^2) \,.\, x = 0$,

(II.) from $t = \pi$ till $t = 2\pi$: $\dfrac{d^2x}{dt^2} + (\omega^2 - \alpha^2) \,.\, x = 0$.

In either of the two intervals the sine and cosine solutions may be immediately written down and we have the boundary

condition between the interval-solutions, that x and $\frac{dx}{dt}$ be continuous at the boundary. These conditions, together with those for $t=0$, enable us to determine the four constants of integration and hence the complete solution.

From this complete solution we easily calculate

$$\cos 2\pi\mu = \tfrac{1}{2}[f(2\pi)+\phi'(2\pi)]$$

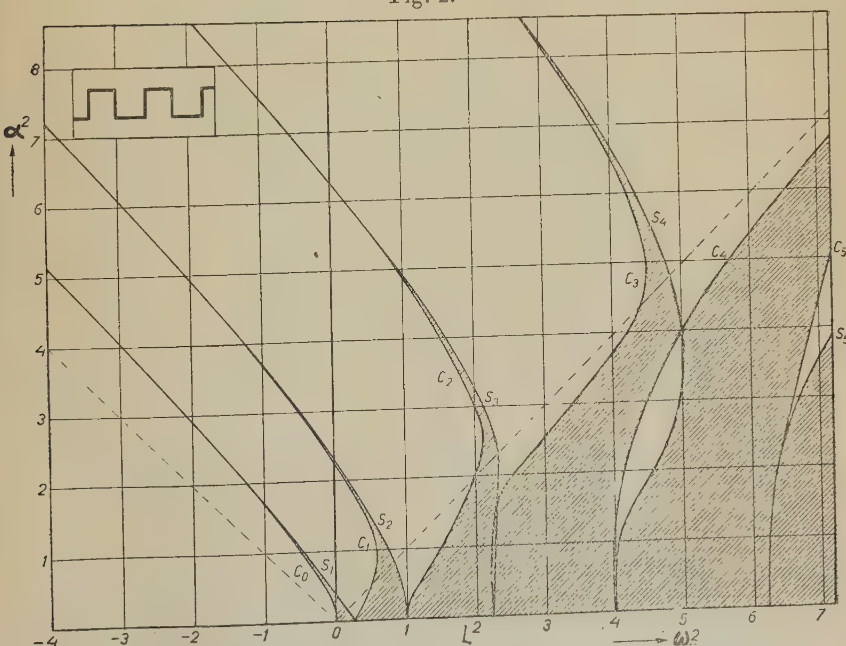
and find :

$$\left.\begin{array}{l}
\text{(I.) for } \omega^2 > \alpha^2 > 0 : \\
\qquad \cos 2\pi\mu = \cos x_1 \cos x_2 - \dfrac{1}{2}\left(\dfrac{x_1}{x_2}+\dfrac{x_2}{x_1}\right)\sin x_1 \sin x_2, \\
\text{(II.) for } \omega^2 < \alpha^2 : \\
\qquad \cos 2\pi\mu = \cos x_1 \cosh x_3 - \dfrac{1}{2}\left(\dfrac{x_1}{x_3}-\dfrac{x_3}{x_1}\right)\sinh x_3 \sin x_1,
\end{array}\right\} \quad \ldots \quad (8)$$

where

$$x_1^2 = \pi^2(\omega^2+\alpha^2),$$
$$x_2^2 = \pi^2(\omega^2-\alpha^2),$$
$$x_3^2 = \pi^2(\alpha^2-\omega^2).$$

Fig. 2.

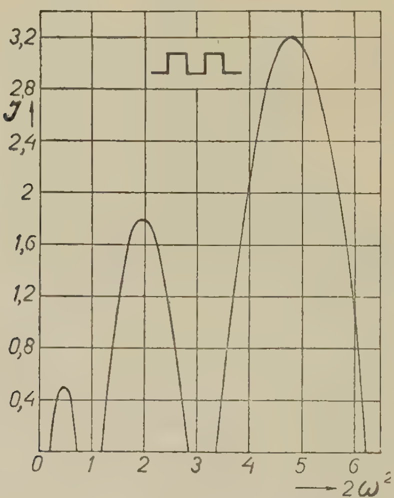


From the equations (8) we may plot curves (fig. 2) in the (α^2, ω^2)-plane, constituting the boundaries between the

regions of unstable (white in fig. 2) and of indifferent (shaded in fig. 2) (stable) movement. We may consider the diagram thus obtained as a plane, and may imagine the value of $I = \cos 2\pi\mu - 1$ plotted perpendicularly on this plane above the regions of unstable movement. Thus a three-dimensional representation is obtained, in which the plane (sea) constitutes the region of indifferent movement and the mountains cover the regions of unstability, the height of them at every point being a measure for the instability at that point.

In fig. 2 *a* a cut through the unstable mountains has been made along the line under 45° in the first quadrant of fig. 2.

Fig. 2 *a*.



Three special cases of equations (8) will now be considered separately.

(I.) *Small ripple* : $\frac{\alpha^2}{\omega^2} \ll 1$.

We easily deduce from the first one of equations (8) :

$$\cos 2\pi\mu = \cos 2\pi\omega - \frac{1}{2}\frac{\alpha^4}{\omega^4}\sin^2 \pi\omega. \quad . \quad . \quad . \quad (9)$$

Thus it is seen that the unstable areas starting from the points ($\alpha^2 = 0$; $\omega = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots$) are of greater instability in the neighbourhood of the ω^2-axis than the unstable regions starting from the points ($\alpha^2 = 0$; $\omega = 1, 2, 3 \ldots$).

(II.) *Fast ripple* :

$$|\alpha^2| << 1,$$
$$|\omega^2| << 1,$$
$$|\alpha^2| > |\omega^2|.$$

In this case we deduce from the second equation (8) :

$$\cos 2\pi\mu = 1 - 2\pi^2\omega^2 - \tfrac{1}{6}\pi^4\alpha^4. \quad . \quad . \quad . \quad (10)$$

Hence, if $\omega^2 > 0$, the movement must be stable; but if $\omega^2 < 0$, we may either have a stable or unstable movement, the boundary between these two regions being given by

$$-\omega^2 = \frac{\pi^2}{12}\alpha^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Thus, the curve C_0 of fig. 2 is given by equation (11) in the neighbourhood of $\alpha^2 = \omega^2 = 0$.

(III.) *Large ripple* (and ω^2 negative) or
Asymptotic behaviour of the boundary-lines for

$$\alpha^2 \geq\!> 1,$$
$$|\omega^2| >> 1.$$

In this case we deduce from the second equation (8) :

$$\frac{\omega^2}{\alpha^2} = \cos \pi \sqrt{\left(1 - \frac{\omega^2}{\alpha^2}\right)} \,.\, \alpha^2. \quad . \quad . \quad . \quad . \quad (12)$$

From equation (12) it follows that all boundary-lines tend asymptotically to straight lines under 45° in the second quadrant of our diagram (fig. 2).

In general, from this diagram we may make the following deductions:—

(I.) The unstable states of motion cover a larger area than the stable ones. This corresponds to the conclusion made from general considerations, that there are more possibilities of unstable movement than of stable movement.

(II.) Below the dotted line under 45° in the first quadrant of fig. 2 the motion is in general stable, the stable areas being cut by relatively small unstable ones. In this case $(\alpha^2 < \omega^2 > 0)$ the ripple (fig. 1) does not touch the zero line, and the coefficient of x (being proportional to the elastic force) in equation (5) remains always positive. *Thus without ripple we should have a stable movement, and the ripple under certain conditions may make the movement unstable.*

(III.) Above and to the left of the 45° line mentioned in the foregoing case, we have, in general, unstable motion, and the unstable areas of fig. 2 are separated by small stable areas. Moreover, the instability increases considerably with

the distance from the aforesaid line, attaining a maximum under 45° in the second quadrant of fig. 2 and in infinity.

To the right of the line $\omega^2=0$ of fig. 2 the motion would without ripple have been of stable type, and thus the instability is exclusively due to the ripple.

(IV.) In the region to the left of the aforesaid line the motion is unstable without ripple, and the ripple has thus the effect of stabilization.

Physically this case may be realized by a reversed pendulum, which can be stabilized by a fast up and down movement of the lower end [(3)]. We may infer from fig. 2 that this stabilization ceases, however, if the ratio of the amplitude of the ripple to the length of the pendulum exceeds a certain magnitude. This has been verified by experiment; a reversed pendulum of length approximately equal to the amplitude of the movement of the support is, with a slow ripple-frequency, stable, but, after increasing the frequency of the ripple, becomes unstable again.

(V.) If ω^2 be negative and $|\omega^2| > \alpha^2$, the coefficient of x in equation (5) is always negative. Thus, the curvature of the solution in the (x, t)-plane being always of one sign, motion cannot be stable, as x increases indefinitely with t. Hence, below the 45° line in the second quadrant of fig. 2 no stable areas can exist.

Now that the general properties of motion represented by equation (5) have been discussed, we shall proceed to consider Mathieu's equation (3).

The analysis leading to equations (6), (7), and (7′) is valid also in this case.

Let, again,

$$\frac{a+d}{2} = \cos 2\pi\mu,$$

then

$$\sigma = e^{\pm j \cdot 2\pi \cdot \mu}; \quad j = \sqrt{-1}.$$

One can easily show that μ can only be real or purely imaginary, but can never be complex.

Thus fig. 3 shows μ, as given by

$$2\pi\mu = \operatorname{arc\,cos}\frac{a+d}{2},$$

$$i.\,e.\text{ when } \left|\frac{a+d}{2}\right| > 1, \quad \text{then } 2\pi j\beta = \operatorname{arc\,cos}\frac{a+d}{2},$$

$$\text{and when } \left|\frac{a+d}{2}\right| < 1, \quad \text{then } 2\pi\alpha = \operatorname{arc\,cos}\frac{a+d}{2},$$

where α and β are plotted as ordinates in fig. 3.

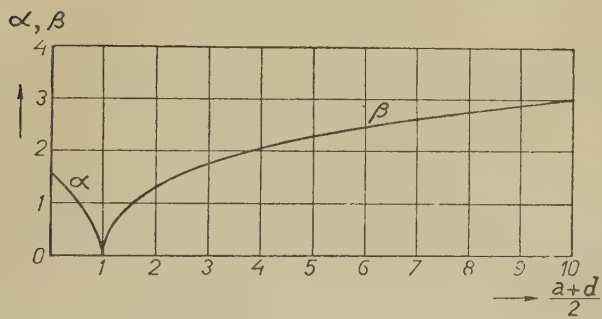
Thus the three cases correspond respectively to

(*a*) $|\cos 2\pi\mu| > 1$ or $\mu =$ imaginary : unstable,

(*b*) $|\cos 2\pi\mu| < 1$ or $\mu =$ real : stable,

(*c*) $|\cos 2\pi\mu| = 1$ or $\mu = 0$ or $\frac{1}{2}$: unstable.

Fig. 3.



In case (*c*) the general solution has no longer the form given by Floquet's theorem, but becomes [16]

$$A \,.\, \Phi(t) + B \,.\, t \,.\, \Psi(t),$$

Φ and Ψ being functions with period 2π in t.

Thus this case generally corresponds to an unstable movement, the movement being, however, of stable type if $B=0$, which depends on the initial conditions.

Equation (3) being a special form of Hill's general equation, we may apply Hill's analysis to equation (3) also.

Two independent solutions of equation (3) may be obtained in the form :

$$e^{j\mu . t} \sum_{-\infty}^{\infty} b_n e^{jnt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and

$$e^{-j\mu t} \sum_{-\infty}^{\infty} b_n e^{-jnt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Substituting (13) or (14) into the differential equation and equating coefficients of equal powers of e to zero, we obtain a determinantal expression, which constitutes an equation for μ :

$$\Delta(\mu) \equiv \left| \begin{array}{l} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \ldots 0, \; \frac{\alpha^2}{2[\omega^2-(\mu+1)^2]}, \; 1, \; \frac{\alpha^2}{2[\omega^2-(\mu+1)^2]}, \; 0, \; 0, \; 0, \; 0 \ldots \; \ldots \\ \ldots \; \ldots, \; 0, \; 0, \; \frac{\alpha^2}{2[\omega^2-\mu^2]}, \; 1, \; \frac{\alpha^2}{2[\omega^2-\mu^2]}, \; 0, \; 0, \; 0 \ldots \; \ldots \\ \ldots \; \ldots, \; 0, \; 0, \; 0, \; \frac{\alpha^2}{2[\omega^2-(\mu+1)^2]}, \; 1, \; \frac{\alpha^2}{2[\omega^2-(\mu+1)^2]}, \; 0, \; 0, \ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array} \right| = 0. \quad . \; . \; . \; (15)$$

Hill has shown that this equation may be written:

$$\sin^2 \pi\mu = \sin^2 \pi\omega \, . \, \Delta(0), \quad . \; . \; . \; . \; . \quad (16)$$

whe

$$\Delta(0) = [\Delta(\mu)]_{\mu=0}.$$

Now, if μ is given a fixed real value, the determinantal equation (15) corresponds to an equation between ω^2 and α^2. Moreover, any real value of μ corresponds to a periodic solution of equation (3).

Thus, for any real value of μ, the expressions (13) and (14), together with the determinantal equation (15) between α^2 and ω^2, are periodic solutions of Mathieu's equation (3), in which ω^2 and α^2 are related by equation (15). The coefficients b_n in the expressions (13) and (14) are determined by the set of equations:

$$[\omega^2-(\mu+n)^2] \, . \, b_n + \frac{\alpha^2}{2} \, . \, (b_{n-1}+b_{n+1}) = 0.$$

We shall call the expressions (13) and (14) for any real μ, defined as above, respectively:

$$\left. \begin{array}{l} \mathrm{M}e^{(\mathrm{I})} = e^{j\mu t} \sum\limits_{-\infty}^{\infty} b_n \, . \, e^{jnt} \\ \text{and} \\ \mathrm{M}e^{(\mathrm{II})} = e^{-j\mu t} \sum\limits_{-\infty}^{\infty} b_n \, . \, e^{-jnt}. \end{array} \right\} \quad . \; . \; . \quad (17)$$

Now, by combining the functions (17), we obtain

$$\begin{aligned} \mathrm{C}e &= \tfrac{1}{2}(\mathrm{M}e^{(\mathrm{I})} + \mathrm{M}e^{(\mathrm{II})}) = \sum_{-\infty}^{\infty} b_n \, . \cos(\mu+n)t \\ &= b_0 + \sum_{1}^{\infty} (b_n + b_{-n}) \, . \cos(n+\mu)t, \quad . \; . \; . \; . \quad (18) \end{aligned}$$

and

$$\begin{aligned} \mathrm{S}e &= \frac{1}{2j}(\mathrm{M}e^{(\mathrm{I})} - \mathrm{M}e^{(\mathrm{II})}) = \sum_{-\infty}^{\infty} b_n \, . \sin(n+\mu)t \\ &= \sum_{1}^{\infty} (b_n - b_{-n}) \sin(n+\mu)t. \quad . \; . \; . \; . \; . \; . \quad (19) \end{aligned}$$

The functions (18) and (19) will further be called *generalized Mathieu Functions.* Corresponding to the fact that the differential equation (3) contains 2 parameters, any Mathieu function will be defined by two numbers—an index and the value of α^2 assigned to the function. The value of ω^2 is then determined from equation (15).

The index is determined by making

$$b_n + b_{-n} = 1$$

in equation (18), thus defining $Ce_{2(n+\mu)}(\alpha^2, t)$, and by making

$$b_n - b_{-n} = 1$$

in equation (19), thus defining $Se_{2(n+\mu)}(\alpha^2, t)$.

Hence, we have the generalized Mathieu functions :

$$Ce_{2(n+\mu)}(\alpha^2, t) \quad \text{and} \quad Se_{2(n+\mu)}(\alpha^2, t). \quad . \quad . \quad (20)$$

If

$$\mu = 0 ; \tfrac{1}{2},$$

we have Mathieu functions of integral degree,

$$Ce_m(\alpha^2, t) \quad \text{and} \quad Se_m(\alpha^2, t), \quad . \quad . \quad . \quad (21)$$

whereas the generalized Mathieu functions (20) may also be of fractional degree.

In the literature most attention has been given to Mathieu functions of integral degree, as these functions occur in several boundary problems in physics (1), but Mathieu functions of fractional degree have only been considered very seldom(17).

According to the definition given above, the determinantal equation (15) for any real value of μ gives a relation between ω^2 and α^2, which is represented by a curve in the (α^2, ω^2)-plane. The curves corresponding to the totality of all real values of μ cover the whole stable (indifferent) motion area of the (α^2, ω^2)-plane. Thus, if we could draw all these curves, we would immediately have solved the first problem which presents itself, viz., the conditions of stable motion.

In any point of one of these curves, for which

$$\mu \neq 0 \text{ or } \tfrac{1}{2},$$

we have at the same time a solution $Ce_{2(n+\mu)}(\alpha^2, t)$ and $Se_{2(n+\mu)}(\alpha^2, t)$, so that the general solution is

$$A \, . \, Ce_{2(n+\mu)}(\alpha^2, t) + B \, . \, Se_{2(n+\mu)}(\alpha^2, t).$$

If

$$\mu = 0 \text{ or } \tfrac{1}{2},$$

the general solution along the curves, represented by equation (15), can no longer be of the aforesaid form, but must be either [13]:

$$A \,.\, Se_m(\alpha^2, t) + B \,.\, \Phi_m(\alpha^2, t),$$

where $\Phi_m(\alpha^2, t)$ is a function which contains t explicitly (before some of the terms with sine) and which thus is non-periodic and even, or

$$A \,.\, Ce_m(\alpha^2, t) + B \,.\, \Psi_m(\alpha^2, t),$$

where Ψ_m is a function of the same character as Φ_m but odd.

Hence, from the points of the ω^2-axis, corresponding to

$$\mu = 0 \text{ or } \tfrac{1}{2},$$

and with the aid of equation (15), *two* curves for constant μ can be drawn, these points being double points.

We can easily see, from equation (16), where these double points are to be found in the (ω^2, α^2)-plane.

The two corresponding values of μ are zero and 1/2. If μ is zero, then for $\alpha^2 \to 0$,

$$\Delta(0) = 1,$$

and we have from (16):

$$\omega = 0, 1, 2, 3 \ldots .$$

Fig. 4.

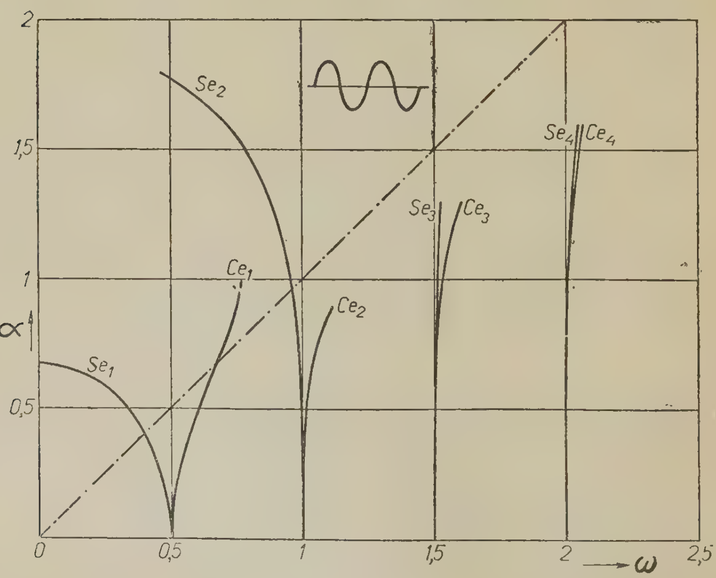


Fig. 5.

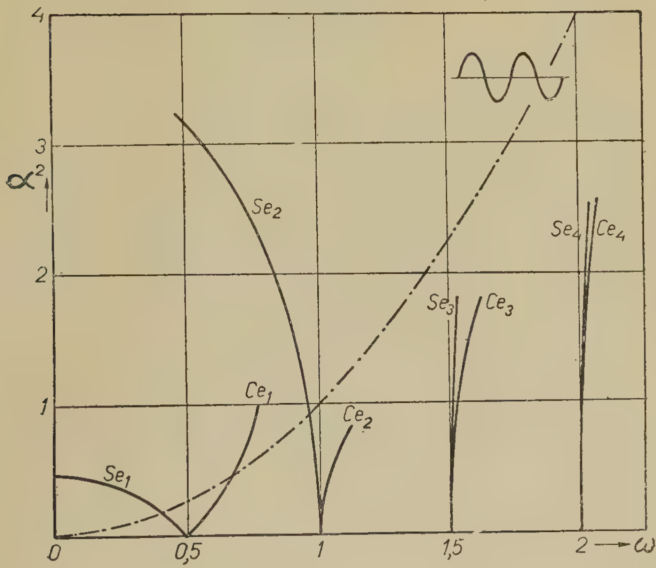

4
3
2
1
α^2
0
0,5
1
1,5
2
ω
Se_1
Ce_1
Se_2
Ce_2
Se_3
Ce_3
Se_4
Ce_4

Fig. 6.

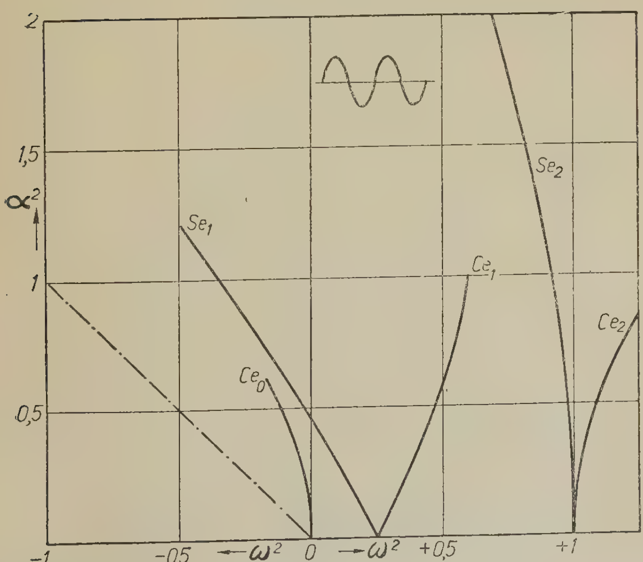

2
1,5
1
0,5
α^2
-1
-0,5
ω^2
0
ω^2
+0,5
+1
Se_1
Ce_0
Ce_1
Se_2
Ce_2

Fig. 7.

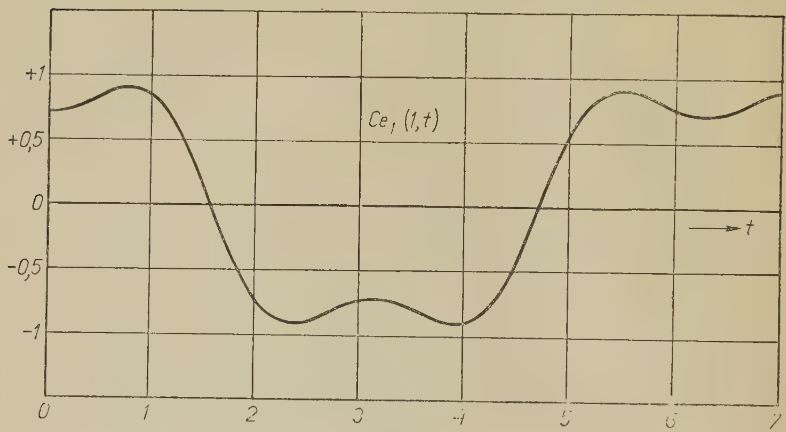
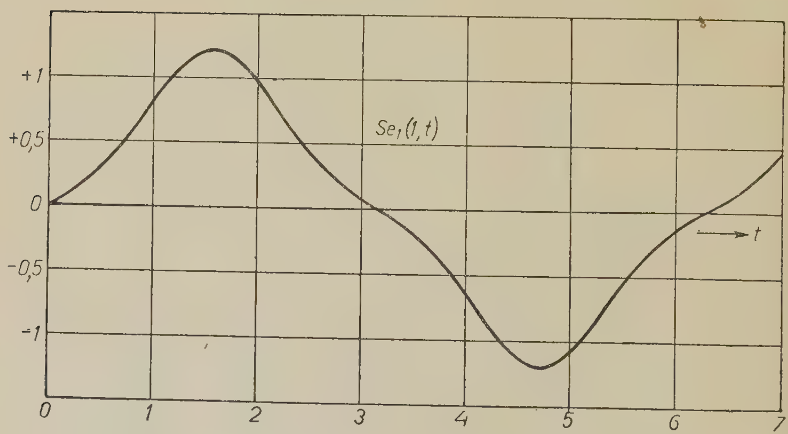


Fig. 8.



If $\mu = 1/2$, and again for $\alpha^2 \to 0$, (16) yields

$$1 = \sin^2 \pi\omega$$

or

$$\omega = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2} \ldots .$$

Numerically, the Mathieu Functions of integral and of fractional order, which reduce to

$$[Ce_m(\alpha^2, t)]_{\alpha^2 \to 0} = \cos \frac{m}{2} t$$

and

$$[Se_m(\alpha^2, t)]_{\alpha^2 \to 0} = \sin \frac{m}{2} t,$$

are easily derived by the method followed * (11) by Mathieu.

We obtain the series (with Mathieu):

$$Ce_1(\alpha^2, t) = \cos \frac{t}{2} - \frac{\alpha^2}{4} \cos \tfrac{3}{2} t + 4\alpha^4 \cdot \left(\frac{\cos \frac{5}{2} t}{192} - \frac{\cos \frac{3}{2} t}{64} \right)$$
$$- 8\alpha^6 \cdot \left(\frac{\cos \frac{7}{2} t}{9216} - \frac{\cos \frac{5}{2} t}{1152} + \frac{\cos \frac{3}{2} t}{1536} \right)$$
$$+ 16\alpha^8 \left(\frac{\cos \frac{9}{2} t}{737280} - \frac{\cos \frac{7}{2} t}{49152} + \frac{\cos \frac{5}{2} t}{24576} + \frac{11 \cos \frac{3}{2} t}{36864} \right) + \dots, \quad . \quad . \quad . \quad (22)$$

with

$$\omega^2 = \frac{1}{4} + \frac{\alpha^2}{2} - \frac{\alpha^4}{8} - \frac{\alpha^6}{32} - \frac{\alpha^8}{384} + \frac{11\alpha^{10}}{4608} + \dots \quad (23)$$

The curve represented by equation (23) has been plotted as (Ce_1) in figs. 4, 5, and 6, and the curve (22) in fig. 7 for $\alpha^2 = 1$.

$$Ce_2(\alpha^2, t) = \cos t + 2\alpha^2 \left(-\frac{\cos 2t}{12} + \frac{1}{4} \right) + \frac{4\alpha^4}{384} \cos 3t$$
$$- 8\alpha^6 \cdot \left(\frac{\cos 4t}{23040} + \frac{43 \cos 2t}{13824} + \frac{5}{192} \right)$$
$$+ 16\alpha^8 \left(\frac{\cos 5t}{2211840} + \frac{287 \cos 3t}{2211840} \right)$$
$$+ 32\alpha^{10} \cdot \left(\frac{-\cos 6t}{309657600} - \frac{41 \cos 4t}{16588800} + \frac{21059 \cos 2t}{79626240} + \frac{1363}{221184} \right) + \dots, \quad . \quad (24)$$

with

$$\omega^2 = 1 + \frac{5}{12} \alpha^4 - \frac{763}{3456} \alpha^8 + \frac{1002419}{4976640} \alpha^{12} + \dots \quad (25)$$

* Mathieu has given series which, as far as we are aware, are best suited for numerical calculations.

The curve given by equation (25) has been plotted as curve Ce_2 in figs. 4, 5, and 6.

$$Ce_3 = \cos\tfrac{3}{2}t + 2\alpha^2 . \left(-\frac{\cos\frac{5}{2}t}{16} + \frac{\cos\frac{t}{2}}{8}\right) + 4\alpha^4\left(\frac{\cos\frac{7}{2}t}{640} + \frac{\cos\frac{t}{2}}{64}\right)$$

$$+ 8\alpha^6 . \left(\frac{-\cos\frac{9}{2}t}{46080} - \frac{7\cos\frac{5}{2}t}{20480} + \frac{\cos\frac{t}{2}}{1024}\right)$$

$$+ 16\alpha^8 . \left(\frac{\cos\frac{11}{2}t}{2^{14} . 3^2 . 5 . 7} + \frac{17\cos\frac{7}{2}t}{2^{15} . 3^2 . 5} - \frac{\cos\frac{5}{2}t}{2^{14}} - \frac{\cos\frac{t}{2}}{2^{12}}\right)$$

$$+ \ldots, \qquad (26)$$

with

$$\omega^2 = \frac{9}{4} + \frac{\alpha^4}{16} + \frac{\alpha^6}{32} + \frac{13\alpha^8}{5120} - \frac{5\alpha^{10}}{2048} + \ldots \qquad (27)$$

The curve given by equation (27) has been plotted as Ce_3 in figs. 4 and 5.

$$Ce_4 = \cos 2t + 2\alpha^2 . \left(-\frac{\cos 3t}{20} + \frac{\cos t}{12}\right) + 4\alpha^4 . \left(\frac{\cos 4t}{960} + \frac{1}{192}\right)$$

$$+ 8\alpha^6\left(\frac{-\cos 5t}{80640} - \frac{13\cos 3t}{96000} + \frac{11\cos t}{17280}\right)$$

$$+ 16\alpha^8 . \left(\frac{\cos 6t}{10321920} + \frac{23\cos 4t}{6048000} - \frac{1}{92160}\right)$$

$$- 32\alpha^{10} . \left(\frac{\cos 7t}{1857945600} + \frac{53\cos 5t}{1032192000} + \frac{4037\cos 3t}{2419200000} + \frac{439\cos t}{62208000}\right) + \ldots, \qquad (28)$$

with

$$\omega^2 = 4 + \frac{\alpha^4}{30} + \frac{433\alpha^8}{216000} - \frac{189983\,\alpha^{12}}{1360800000} + \ldots \qquad (29)$$

The curve given by (29) has been drawn as curve Ce_4 in figs. 4 and 5.

$$Se_1 = \sin\frac{t}{2} - \frac{\alpha^2}{4}\sin 3\,\frac{t}{2} + 4\alpha^4 . \left(\frac{\sin\frac{5}{2}t}{192} + \frac{\sin\frac{3}{2}t}{64}\right)$$

$$- 8\alpha^6 . \left(\frac{\sin\frac{7}{2}t}{9216} + \frac{\sin\frac{5}{2}t}{1152} + \frac{\sin\frac{3}{2}t}{1536}\right)$$

$$+ 16\alpha^8 . \left(\frac{\sin\frac{9}{2}t}{737280} + \frac{\sin\frac{7}{2}t}{49152} + \frac{\sin\frac{5}{2}t}{24576} - \frac{11\sin\frac{3}{2}t}{36864}\right) + \ldots, \qquad (30)$$

with

$$\omega^2 = \frac{1}{4} - \frac{\alpha^2}{2} - \frac{\alpha^4}{8} + \frac{\alpha^6}{32} \quad \frac{\alpha^8}{384} - \frac{11\alpha^{10}}{4608} + \ldots \qquad (31)$$

The curve (31) is found as curve Se_1 in figs. 4, 5, 6, and the curve (30) for $\alpha^2=1$ in fig. 8.

$$Se_2 = \sin t - \frac{\alpha^2}{6}\sin 2t + \frac{\alpha^4}{96}.\sin 3t$$

$$+8\alpha^6.\left(\frac{-\sin 4t}{23040}+\frac{5\sin 2t}{13824}\right)$$

$$+16\alpha^8.\left(\frac{\sin 5t}{2211840}+\frac{37\sin 3t}{2209140}\right)$$

$$+32\alpha^{10}.\left(\frac{-\sin 6t}{309657600}+\frac{11\sin 4t}{33177600}-\frac{289\sin 2t}{79626240}\right)+\dots, \quad . \quad . \quad . \quad (39)$$

with

$$\omega^2 = 1-\frac{\alpha^4}{12}+\frac{5\alpha^8}{3456}-\frac{289\,\alpha^{12}}{4976640}+\dots\dots \quad . \quad (33)$$

The curve (33) is drawn as curve Se_2 on figs. 4, 5, and 6.

$$Se_3 = \sin 3\frac{t}{2} + 2\alpha^2.\left(-\frac{\sin\frac{5}{2}t}{16}+\frac{\sin\frac{t}{2}}{8}\right)+4\alpha^4.\left(\frac{\sin\frac{7}{2}t}{640}-\frac{\sin\frac{t}{2}}{64}\right)$$

$$+8\alpha^6.\left(\frac{-\sin\frac{9}{2}t}{46080}-\frac{7\sin\frac{5}{2}t}{20480}+\frac{\sin\frac{t}{2}}{1024}\right)$$

$$+16\alpha^8.\left(\frac{\sin\frac{11}{2}t}{2^{14}.3^2.5.7}+\frac{17\sin\frac{7}{2}t}{2^{15}.3^2.5}+\frac{\sin\frac{5}{2}t}{2^{14}}+\frac{\sin\frac{t}{2}}{2^{12}}\right)$$

$$+\dots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

with

$$\omega^2 = \frac{9}{4}+\frac{\alpha^4}{16}-\frac{\alpha^6}{32}+\frac{13\alpha^8}{5120}+\frac{5\alpha^{10}}{2048}+\dots.$$

The curve (35) is drawn as curve Se_3 in figs. 4 and 5.

$$Se_4 = \sin 2t + 2\alpha^2.\left(-\frac{\sin 3t}{20}+\frac{\sin 2t}{12}\right)+\frac{\alpha^4}{240}\sin 4t$$

$$-8\alpha^6.\left(\frac{\sin 5t}{80640}+\frac{13\sin 3t}{96000}-\frac{\sin t}{4320}\right)$$

$$+16\alpha^8.\left(\frac{\sin 6t}{10321920}+\frac{23\sin 4t}{6048000}\right)$$

$$+32\alpha^{10}.\left(\frac{-\sin 7t}{1857945600}-\frac{53\sin 5t}{1032192000}+\frac{293\sin 3t}{2419200000}\right.$$

$$\left.+\frac{397\sin t}{124416000}\right)+\dots, \quad . \quad (36)$$

with

$$\omega^2 = 4 + \frac{\alpha^4}{30} - \frac{317}{216000}\alpha^8 + \frac{4507\,\alpha^{12}}{85050000} + \dots \quad (37)$$

The curve given by (37) has been drawn as Se_4 in figs. 4 and 5.

The series of Se can be obtained from those of Ce by anging $\frac{t}{2}$ in $\frac{\pi}{2} - \frac{t}{2}$ and α^2 in $-\alpha^2$.

In general, we have:

$$\begin{aligned}
Ce_g = \cos\frac{g}{2}t &+ 2\alpha^2 . \left[\frac{-\cos(g+2).\frac{t}{2}}{4(g+1)} + \frac{\cos(g-2).\frac{t}{2}}{4(g-1)}\right] \\
&+ 4\alpha^4\left[\frac{\cos(g+4)\frac{t}{2}}{32(g+1)(g+2)} + \frac{\cos(g-4)\frac{t}{2}}{32(g-1)(g-2)}\right] \\
&+ 8\alpha^6 . \left[\frac{-\cos(g+6)\frac{t}{2}}{2^7.3.(g+1)(g+2).(g+3)}\right. \\
&\qquad - \frac{(g^2+4g+7)\cos(g+2)\frac{t}{2}}{2^7.(g+1)^3.(g-1).(g+2)} \\
&\qquad + \frac{(g^2-4g+7)\cos(g-2)\frac{t}{2}}{2^7.(g-1)^3.(g+1).(g-2)} \\
&\qquad \left. + \frac{\cos(g-6)\frac{t}{2}}{2^7.3.(g-1)(g-2)(g-3)}\right] \\
&+ 16\alpha^8 . \left[\frac{\cos(g+8)\frac{t}{2}}{2^{11}.3.(g+1).(g+2)(g+3)(g+4)}\right. \\
&\qquad + \frac{(g^3+7g^2+20g+20).\cos(g+4)\frac{t}{2}}{2^8.3.(g+1)^3.(g-1)(g+2)^2.(g+3)} \\
&\qquad + \frac{(g^3-7g^2+20g-20).\cos(g-4)\frac{t}{2}}{2^8.3.(g-1)^3.(g+1).(g-2)^2.(g-3)} \\
&\qquad \left. + \frac{\cos(g-8)\frac{t}{2}}{2^{11}.3.(g-1)(g-2)(g-3).(g-4)}\right] \\
&+ \dots, \qquad (38)
\end{aligned}$$

with

$$\begin{aligned}
\omega^2 = \frac{g^2}{4} &+ \frac{4\alpha^4}{2.(g^2-1)} + \frac{(5g^2+7)16\alpha^8}{32(g^2-1)^3.(g^2-4)} \\
&+ \frac{(9g^6+22g^4-203g^2-116)}{2(g^2-1)^5.(g^2-4)^2.(g^2-9)}.\alpha^{12} + . \quad (39)
\end{aligned}$$

Formulæ (38) and (39) may be applied whenever $g \neq 1, 2, 3,$ or 4.

It is easily seen that the formulæ for Ce are obtained from those for Se by changing α^2 into $-\alpha^2$ and $\frac{t}{2}$ into $\frac{\pi}{2} - \frac{t}{2}$.

Furthermore, from the general formula for ω^2 it is obvious that as long as g is not an integer the curve in the ω^2, α^2-plane corresponding to Se_g is the same as that corresponding to Ce_g. Only if g is equal to an integer the two curves separate, including between them an area of instability.

Considering figs. 4, 5, and 6 generally and comparing them with fig. 2, it is seen that *the general character of the unstable regions is the same in the case of a sinusoidal ripple as in the case of a rectangular ripple.*

Mathieu functions of integer order are orthogonal, viz.:

$$\int_0^{4\pi} \mathrm{Ce}_g \,.\, \mathrm{Se}_k \,.\, dt = 0,$$

$$\int_0^{4\pi} \mathrm{Ce}_g \,.\, \mathrm{Ce}_k \,.\, dt \begin{array}{l} \neq 0 \text{ if } k = g, \\ = 0 \text{ if } k \neq g, \end{array}$$

$$\int_0^{4\pi} \mathrm{Se}_g \,.\, \mathrm{Se}_k \,.\, dt \begin{array}{l} \neq 0 \text{ if } k = g, \\ = 0 \text{ if } k \neq g. \end{array}$$

Mathieu functions of fractional degree have the property of orthogonality in a similar manner:

$$\int_0^{4\pi/s} \mathrm{Ce}_s \,.\, \mathrm{Se}_{k+s} \,.\, dt = 0, \quad \begin{pmatrix} k = \text{integer} \\ s = \text{fraction} \end{pmatrix}$$

$$\int_0^{4\pi/s} \mathrm{Ce}_s \,.\, \mathrm{Ce}_{s+k} \,.\, dt \begin{array}{l} \neq 0 \text{ if } k = 0, \\ = 0 \text{ if } k \neq 0, \end{array}$$

$$\int_0^{4/\pi s} \mathrm{Se}_s \,.\, \mathrm{Se}_{s+k} \,.\, dt \begin{array}{l} \neq 0 \text{ if } k = 0, \\ = 0 \text{ if } k \neq 0. \end{array}$$

Similar relations have already been obtained by E. G. Poole [17].

Mathieu functions enable us to solve the Mathieu equation in any region of stable movement. In the regions of unstable movement, however, they are no longer valid.

A method for obtaining the value of μ and the solution of equation (3) in these regions has been given by E. T. Whittaker [1]. The series given by him ends with α^8 and so enables us to find the solution numerically for about the same values of α as the series of Mathieu given above.

Various other methods of attack have been given for

equation (3), but the methods mentioned above have been the most successful numerically hitherto.

From the similarity of the diagrams representing the regions of stable and unstable movement of equations (3) and (5), we may expect that all the boundary lines between these regions also for equation (2) cut the α^2-axis for large values of α^2 at points corresponding to

$$\alpha = n + \tfrac{1}{2},$$

where $n = 1, 2, 3$, etc.

Finally, these boundary lines tend to straight lines under 45° in the second quadrant of fig. 2, the stable regions being rather large *below* the 45° line in the first quadrant and very small relatively to the unstable regions *above* and to the left of the aforesaid line. Thus there seems to be little doubt that the general behaviour of the solutions of the Mathieu equation will be similar to the behaviour, fully described above, of the case where the ripple is rectangular instead of sinusoidal.

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III. *On the Calculation of the Periods of Circular Membranes and Disks.* *By* R. C. J. Howland, *M.A.*, *M.Sc.*, *University College, London* *.

Introductory.

THE theory of the vibrations of a uniform circular membrane was given by Poisson † and Clebsch ‡. The vibrations of a circular disk were considered by Kirchhoff §. Both theories are fully expounded by Rayleigh ||, who gives also a method for calculating the effect on the periods of slight deviations from uniformity ¶.

The vibrations of some types of non-uniform membranes have been discussed by Routh **, using the method of conformal transformation. A solution in Bessel functions for the symmetrical modes of a type of loaded membrane has been given by R. N. Ghosh ††. For a special kind of non-uniform disk, solutions in power series are obtainable ‡‡. Apart from these special cases, only approximate methods have been used for determining the periods of non-uniform membranes and disks. Rayleigh's method of an assumed mode §§ has been applied successfully to disks |||| and to spinning disks ¶¶, and the Ritz method of minimum energy *** may also be used. It is, however, not always easy to find a suitable assumed mode, and a wrong assumption may lead to very large errors †††.

In a recent paper by the present writer ‡‡‡ it was shown that very good values for the periods of bars may be obtained through the medium of an integral equation §§§.

* Communicated by the Author.

† *Mém. de l'Académie*, viii. (1829).

‡ 'Theorie der Elasticität fester Körper' (1862).

§ *Crelle*, xl. p. 51 (1850).

|| 'Theory of Sound,' chapters ix. and x. (2nd ed. revised 1926).

¶ *Ibid.* pars. 208 and 221.

** "Some Applications and Conjugate Functions," Proc. Lond. Math. Soc. xii. p. 73 (1881).

†† "On Indian Drums," Phil. Mag. ser. 6, xlv. pp. 315–316.

‡‡ Prescott, 'Applied Electricity' (London, 1924), par. 338.

§§ 'Theory of Sound,' par. 88.

|||| Prescott, *loc. cit.* pars. 340 *et seq.*

¶¶ Lamb and Southwell, Proc. Roy. Soc. A, vol. xcix. pp. 272–280; and Southwell, *ibid.* vol. ci. pp. 133–152. Also Stodola, *Schweiz. Bauz.* vol. lxiii. pp. 251, 271 (1914).

*** Ritz, *Crelle*, 1903, or *Ges. Werke*, Paris, 1913.

††† *Cf.* the paper by Southwell cited above.

‡‡‡ Phil. Mag. ser. 7, iii. pp. 674–694 (1927).

§§§ A different method of treating the integral equation is given by Schwerin, *Zeit. Tech. Phys.* viii. p. 264 (1927).

It is here shown that the same method may be applied to find the periods of circular membranes and disks. It is theoretically capable of any required degree of accuracy, but it will be seen that, for the higher modes, accuracy can only be attained by laborious calculations.

The Integral Equation.

The membranes and disks considered may be of variable thickness, density, and rigidity. The vibrations may be controlled by either rigidity or tension or by both acting together, and the method of support is indifferent. It is supposed, however, that the density, rigidity, and tension, if variable, are functions only of the distance from the centre, *i. e.* there is circular symmetry. This is to be true also of the mode of support, but need not be true of the mode of vibration.

Let (r, θ) be polar coordinates in the plane of the membrane or disk, the pole being at the centre. On a circle of radius α apply a transverse force of magnitude $\cos n\theta$ per unit length, n being an integer. The deflexion due to this at any point (r, θ) will be written

$$\mathrm{K}_n(r, \alpha) \cos n\theta,$$

where $\mathrm{K}_n(r, \alpha)$ is a function whose form depends on the density, rigidity, tension, and mode of support.

In a vibration with n nodal diameters, the frequency being λ, the deflexion may be written

$$w = w_n(r) \cos n\theta \cos 2\pi\lambda t,$$

and the mass-acceleration per unit length of a ring of mean radius α and breadth $\delta\alpha$ is

$$-4\pi^2\lambda^2 m(\alpha) \,.\, w_n(\alpha) \,.\, \cos n\theta \cos 2\pi\lambda t \,.\, \delta\alpha,$$

m, the mass per unit area, being also a function of the radius vector only.

It follows from d'Alembert's Principle that, if the mass accelerations are reversed and regarded as static forces, they will produce a deflexion identical with the actual instantaneous deflexion. The deflexion due to the reversed mass-accelerations on the ring of breadth $\delta\alpha$ is

$$4\pi^2\lambda^2 m(\alpha) \,.\, w_n(\alpha) \,.\, \cos n\theta \,.\, \cos 2\pi\lambda t \,.\, \delta\alpha \,.\, \mathrm{K}_n(r, \alpha).$$

Integrating this and equating the result to the actual

deflexion, we have, after removing the common factors,

$$w_n(r)=4\pi^2\lambda^2\int_0^a m(\alpha)w_n(\alpha)K_n(r,\alpha)\,d\alpha, \quad . \quad . \quad (1)$$

in which a is the radius of the boundary *.

The Approximate Equation for the Frequencies.

Applying to the integral in (1) the trapezoidal rule with ν intervals, and remembering that, from its definition, $K_0(r\,\alpha)$ is identically zero, we obtain the linear equation

$$w_n(r)=\frac{4\pi^2\lambda^2 a}{\nu}\sum_{q=1}^{\nu}\beta_q m\left(\frac{qa}{\nu}\right)w_n\left(\frac{qa}{\nu}\right)K_n\left(r,\frac{qa}{\nu}\right), \quad . \quad (2)$$

where

$$\left.\begin{aligned}\beta_q&=1, \quad q\neq\nu,\\ \beta_\nu&=\tfrac{1}{2}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In (2) let r take successively the values

$$\frac{a}{\nu},\quad \frac{2a}{\nu},\ \ldots\ldots\ \frac{\nu a}{\nu},$$

and write

$$\left.\begin{aligned}w_n\left(\frac{qa}{\nu}\right)&=w_q, \quad m\left(\frac{qa}{\nu}\right)=m_q,\\ K_n\left(\frac{pa}{\nu},\frac{qa}{\nu}\right)&=k_{pq}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Then

$$w_p=\frac{4\pi^2\lambda^2 a}{\nu}\sum_{q=1}^{\nu}\beta_q m_q k_{pq} w_q, \quad . \quad . \quad . \quad . \quad (5)$$

$$p=1,2,3,\ldots\ldots\nu.$$

Eliminating the w's, we have for λ^2 the equation of degree ν:

$$\begin{vmatrix} m_1k_{11}-\mu, & m_2k_{12}, & \ldots\ldots, & \tfrac{1}{2}m_\nu k_{1\nu} \\ m_1k_{21}, & m_2k_{22}-\mu, & \ldots\ldots, & \tfrac{1}{2}m_\nu k_{2\nu} \\ \cdot & \cdot & \cdot & \cdot \\ m_1k_{\nu1}, & m_2k_{\nu2}, & \ldots\ldots, & \tfrac{1}{2}m_\nu k_{\nu\nu}-\mu \end{vmatrix}=0, \quad (6)$$

where

$$\mu=\frac{\nu}{4\pi^2\lambda^2 a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

When the rim is fixed, the last column of the determinant is missing and the degree of the equation reduces to $\nu-1$.

* For the theory of this type of equation see Whittaker and Watson, 'Modern Analysis,' 3rd ed., par. 11·23.

Determination of $K_n(r, \alpha)$ *for the Membrane.*

The deflexion of a stretched membrane under a given system of transverse force depends only on the tension. It is therefore possible to give general formulæ, independent of the variations in density or thickness of the material. These will enter into the equation for the periods only through the factors m_g.

If the membrane is stretched to a tension T and is acted upon by a transverse force of magnitude $P\cos n\theta$ per unit area, P being any function of r, the deflexion w is given by the equation

$$\nabla^2 w + \frac{P}{T}\cos n\theta = 0,$$

or, writing

$$w = w_n \cos n\theta,$$

$$\frac{d^2 w_n}{dr^2} + \frac{1}{r}\frac{dw_n}{dr} - \frac{n^2}{r^2}w_n = -\frac{P}{T},$$

which may conveniently be written as

$$\frac{1}{r^{n+1}}\frac{d}{dr}\left\{r^{2n+1}\frac{d}{dr}\left(\frac{w_n}{r^n}\right)\right\} = -\frac{P}{T}. \quad . \quad . \quad . \quad (8)$$

To identify w_n with $K_n(r, \alpha)$, let P be zero except when $r=\alpha$, while

$$\lim_{\epsilon \to 0}\int_{\alpha-\epsilon}^{\alpha+\epsilon} P\, dr = 1.$$

Then

$$\begin{aligned}\int_0^r P r^{n+1} dr &= 0 \qquad \text{if} \quad r<\alpha, \\ &= \alpha^{n+1} \quad \text{if} \quad r>\alpha,\end{aligned}$$

and a first integration of (8) gives

$$\frac{d}{dr}\left(\frac{w_n}{r^n}\right) = \frac{A}{r^{2n+1}}\ \Bigg|\ -\frac{\alpha^{n+1}}{Tr^{2n+1}}.$$

In this equation and in all that follows the terms to the left of the vertical bar apply to the whole membrane; those to the right of the bar apply only for $r > \alpha$.

Integrating again,

$$\frac{w_n}{r^n} = \frac{A'}{r^{2n}} + B\ \Bigg|\ -\frac{\alpha^{n+1}}{T}\int_a^r \frac{dr}{r^{2n+1}},$$

or

$$w_n = \frac{A'}{r^n} + Br^n\ \Bigg|\ -\frac{r^{2n}-\alpha^{2n}}{2nT\alpha^{n-1}r^n}.$$

If the membrane is complete, A′ must be zero. If it is fixed at the rim, we have also

$$B=\frac{a^{2n}-\alpha^{2n}}{2nT\alpha^{n-1}a^{2n}}.$$

Hence

$$\left.\begin{aligned}K_n(r,\alpha)&=\frac{a^{2n}-\alpha^{2n}}{2nTa^{2n}\alpha^{n-1}}\cdot r^n,\ r<\alpha,\\ K_n(r,\alpha)&=\frac{a^{2n}-r^{2n}}{2nTa^{2n}r^n}\cdot\alpha^{n+1},\ r>\alpha.\end{aligned}\right\}\quad . \quad . \quad . \quad (9)$$

In the above integration it has been assumed that n is not zero. The formulæ for $n=0$ are

$$\left.\begin{aligned}K_0(r,\alpha)&=\frac{\alpha}{T}\log\frac{a}{\alpha},\ r<\alpha,\\ K_0(r,\alpha)&=\frac{\alpha}{T}\log\frac{a}{r},\ r>\alpha.\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad (10)$$

Calculation of $K_n(r,\alpha)$ *for the Disk.*

If $2h=$ thickness of disk,
$I=2h^3/3$,
$E=$ Young's Modulus,
$\sigma=$ Poisson's Ratio,
$E'=E/(1-\sigma)$,

the deflexion due to a transverse force of magnitude p per unit area is determined by the equation

$$\nabla^2(I\nabla^2 w)=\frac{p}{E'}.\quad . \quad . \quad . \quad . \quad . \quad (11)^*$$

From the general assumption of symmetry, I is a function of r but not of θ; E′ will be supposed constant. Writing $p=P\cos n\theta$, where P has the same meaning as before, we have

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)I\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)w_n=\frac{P}{E'}.\quad (12)$$

Now, if χ is any function of r, we may write

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)\chi=\frac{1}{r^{n+1}}\frac{d}{dr}\left\{r^{n+1}\left(\frac{d\chi}{dr}-n\frac{\chi}{r}\right)\right\}.\quad (13)$$

and

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)\chi=r^{n-1}\frac{d}{dr}\left\{\frac{1}{r^{n-1}}\left(\frac{d\chi}{dr}+n\frac{\chi}{r}\right)\right\}.\quad (14)$$

* Prescott, 'Applied Elasticity,' p. 394.

Putting

$$\chi = \mathrm{I}\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)w_n,$$

we therefore obtain the following two first integrals of (12),

$$\frac{d\chi}{dr}-n\frac{\chi}{r}=\frac{\mathrm{A}}{r^{n+1}}\,\bigg|\,+\frac{1}{\mathrm{E}'}\left(\frac{\alpha}{r}\right)^{n+1},$$

$$\frac{d\chi}{dr}+n\frac{\chi}{r}=\mathrm{B}r^{n-1}\,\bigg|\,+\frac{1}{\mathrm{E}'}\left(\frac{r}{\alpha}\right)^{n-1},$$

whence, by subtraction,

$$2n\frac{\mathrm{I}}{r}\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)w_n=\mathrm{B}r^{n-1}$$
$$-\frac{\mathrm{A}}{r^{n+1}}\,\bigg|\,+\frac{1}{\mathrm{E}'}\left\{\left(\frac{r}{\alpha}\right)^{n-1}-\left(\frac{\alpha}{r}\right)^{n+1}\right\}.$$

Again applying (13) and (14) to obtain first integrals, we find

$$2n\left(\frac{dw_n}{dr}-n\frac{w_n}{r}\right)=\frac{\mathrm{C}}{r^{n+1}}+\frac{\mathrm{B}}{r^{n+1}}\int^r\frac{r^{2n+1}}{\mathrm{I}}dr$$
$$-\frac{\mathrm{A}}{r^{n+1}}\int^r\frac{r}{\mathrm{I}}dr\,\bigg|\,+\frac{1}{\mathrm{E}'r^{n+1}}\int_\alpha^r\left(\frac{r^{2n+1}}{\alpha^{n-1}}-\alpha^{n+1}r\right)\frac{dr}{\mathrm{I}}$$

and

$$2n\left(\frac{dw_n}{dr}+n\frac{w_n}{r}\right)=\mathrm{D}r^{n-1}+\mathrm{B}r^{n-1}\int^r\frac{r}{\mathrm{I}}dr$$
$$-\mathrm{A}r^{n-1}\int^r\frac{dr}{\mathrm{I}r^{2n-1}}\,\bigg|\,+\frac{r^{n-1}}{\mathrm{E}'}\int_\alpha^r\left(\frac{r}{\alpha^{n-1}}-\frac{\alpha^{n+1}}{r^{2n-1}}\right)\frac{dr}{\mathrm{I}}.$$

Again subtracting,

$$4n^2w_n=\mathrm{D}r^n-\frac{\mathrm{C}}{r^n}+\mathrm{B}\left(r^n\int^r\frac{r}{\mathrm{I}}dr-\frac{1}{r^n}\int^r\frac{r^{2n+1}}{\mathrm{I}}dr\right)$$
$$+\mathrm{A}\left(\frac{1}{r^n}\int^r\frac{r}{\mathrm{I}}dr-r^n\int^r\frac{dr}{\mathrm{I}r^{2n-1}}\right)\,\bigg|\,+\frac{1}{\mathrm{E}'}\Bigg[r^n\int_\alpha^r\Bigg(\frac{r}{\alpha^{n-1}}$$
$$-\frac{\alpha^{n+1}}{r^{2n-1}}\Bigg)\frac{dr}{\mathrm{I}}-\frac{1}{r^n}\int_\alpha^r\left(\frac{r^{2n+1}}{\alpha^{n-1}}-\alpha^{n+1}r\right)\frac{dr}{\mathrm{I}}\Bigg]. \quad . \quad (15)$$

When $n=0$, this result must be modified, since (14) now becomes identical with (13) and must be replaced by

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\chi=\frac{1}{r\log r}\frac{d}{dr}\left\{r\log r\frac{d\chi}{dr}-\chi\right\} \quad . \quad (16)$$

Carrying out the integration as before, we get in place of (15)

$$w_0 = \mathrm{A}\left(\log r\int^r \frac{r\log r}{\mathrm{I}}dr - \int^r \frac{r(\log r)^2}{\mathrm{I}}dr\right) + \mathrm{B}\left(\log r\int^r \frac{r}{\mathrm{I}}dr - \int^r \frac{r\log r}{\mathrm{I}}dr\right) + \mathrm{C}\log r + \mathrm{D}\,\Big| + \frac{\alpha}{\mathrm{E}'}\left[\log r\int_\alpha^r \frac{r}{\mathrm{I}}\log\frac{r}{\alpha}dr - \int_\alpha^r \frac{r}{\mathrm{I}}\log r\log\frac{r}{\alpha}dr\right]. \quad . \quad . \quad . \quad . \quad (17)$$

From (15) and (17), when I is known as a function of r, the form of $\mathrm{K}_n(r\alpha)$ corresponding to any given boundary conditions may be determined.

Numerical Results for Uniform Membrane.

The degree of accuracy attainable by the method outlined above is conveniently illustrated by using the method to calculate the periods of a uniform membrane. The method of calculation is sufficiently obvious, and only the results will be given. In Table I., n is the number of nodal diameters, s the number of nodal circles of the mode of vibration. Five of the periods have been calculated with ν, the number of intervals, varying from 4 to 6. The correct values of the periods are added for comparison. When $\nu=6$ the equation for μ is of degree 5; the calculation is then a little long. It will be seen that, for the mode having both a nodal diameter and a nodal circle, more than six intervals would be required to give an accurate result. With $\nu=8$ or 9 the labour of calculation would be very great, but not prohibitive.

The values obtained are all too small. The errors increase, for a constant value of ν, both with s and with n.

Table I

Values of $\lambda\sqrt{\mathrm{T}/ma^2}$.

		$\nu=4$.	$\nu=5$.	$\nu=6$.	True value.
$n=0$ ...	$s=0$	0·38	0·38	0·38	0·38
	$s=1$	0·81	0·85	0·86	0·88
$n=1$ ...	$s=0$	0·59	0·595	0·60	0·61
	$s=1$	0·98	1·03	1·06	1·12
$n=2$ ...	$s=0$	0·75	0·78	0·79	0·82

Results for Uniform Clamped Disk.

For a uniform disk clamped at the rim and of radius a, (17) will be found to lead to

$$\left.\begin{aligned} K_0(r,\alpha) &= \frac{\alpha}{8E'I}\left\{\frac{(a^2-\alpha^2)(a^2+r^2)}{a^2} - 2(r^2+\alpha^2)\log\frac{a}{\alpha}\right\}, \quad r<\alpha, \\ K_0(r,\alpha) &= \frac{\alpha}{8E'I}\left\{\frac{(a^2+\alpha^2)(a^2-r^2)}{a^2} - 2(r^2+\alpha^2)\log\frac{a}{r}\right\}, \quad r>\alpha. \end{aligned}\right\} \quad . \quad . \quad . \quad (18)$$

From these and (6) it is easy to calculate the fundamental frequency. Using four intervals, we find

$$\lambda = 1{\cdot}67\sqrt{E'Ima^4}.$$

With six intervals the coefficient is 1·64. The true value is 1·63.

The convergence to the true value is here rather slower than for the membrane, and this will be found to be the case in general. The accurate calculation of the higher frequencies therefore requires a large number of intervals. But this is compensated by the fact that the modes with nodal circles have much higher frequencies than the corresponding modes with only diametral nodes, and are, in consequence, seldom of importance. If only the modes without nodal circles are to be calculated, it is only necessary to find the highest root of equation (6), while it is known that the other roots are comparatively small. Under these circumstances it is unnecessary to calculate all the coefficients.

Consider, for example, a vibration with one nodal diameter. From (15) we obtain

$$\left.\begin{aligned} K_1(r,\alpha) &= \frac{r}{4E'I}\left[\alpha^2\log\frac{a}{\alpha} - \frac{a^2-\alpha^2}{4a^4}\left\{2a^2\alpha^2+(a^2-\alpha^2)r^2\right\}\right], \\ &\qquad r<\alpha, \\ K_1(r,\alpha) &= \frac{\alpha^2}{4E'Ir}\left[r^2\log\frac{a}{r} - \frac{(a^2-r^2)}{4a^4}\left\{2a^2r^2+(a^2-r^2)\alpha^2\right\}\right], \\ &\qquad r>\alpha. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (19)$$

With five intervals, and writing $\mu = \dfrac{2500\,EI'}{4\pi^2\lambda^2ma^4}$, a slight

modification of (7), the equation for μ becomes

$$\begin{vmatrix} 0{\cdot}8990-\mu, & 1{\cdot}8088, & 1{\cdot}6151, & 0{\cdot}6578 \\ 0{\cdot}4522, & 2{\cdot}5592-\mu, & 2{\cdot}6158, & 1{\cdot}1212 \\ 0{\cdot}1795, & 1{\cdot}1626, & 2{\cdot}3877-\mu, & 1{\cdot}1958 \\ 0{\cdot}0411, & 0{\cdot}2803, & 0{\cdot}6726, & 0{\cdot}6872-\mu \end{vmatrix} = 0$$

or $$\mu^4 - 6{\cdot}533\,\mu^3 - 7{\cdot}483\,\mu^2 - 4{\cdot}30\,\mu - 0{\cdot}61 = 0. \quad (20)$$

Of this equation the highest root is 5·262, leading to $\lambda = 3{\cdot}47\sqrt{\mathrm{EI}/ma^4}$. The correct coefficient is 3·38. If (20) is deprived of its last term and the highest root is again calculated, the result is 5·266 and the value of λ is scarcely altered. If the last two terms of (20) are removed, the higher root of the resulting quadratic is 5·051, which leads to a coefficient 3·52. The approximation has been worsened by only $1\frac{1}{2}$ per cent. Now, even with ν as high as 8 or 10, it is a simple matter to calculate the first three terms in the equation. The result is a quadratic from which the value of λ will usually be obtainable within about 2 per cent.*

IV. *On the Practical Application of the Theory of Vibrations to Systems with several Degrees of Freedom.* By C. Richard Soderberg, *Research Engineer, Westinghouse Electric & Mfg. Co., Power Eng. Dept., East Pittsburgh, Pa., U.S.A.*†

I. Introduction.

There are numerous engineering problems to which the general theory of vibrations can be applied profitably, but the rigorous treatment involves somewhat laborious arithmetical calculations, and is therefore often replaced by more or less unreliable methods of approximation. The author has in mind, particularly, the frequently occurring problems of determining the whirling speeds in multiple bearing machines, the critical speeds for torsional oscillations in line shafting with several masses, and various problems of similar nature encountered in foundation studies.

Ordinarily these problems are treated with sufficient accuracy by the determination of an approximate value for

* For further examples of calculations of this kind see the Author's paper cited above and also a paper entitled "The Vibrations of Frames," Selected Eng. Papers, No. 47, Inst. C.E., 1927.

† Communicated by R. V. Southwell, M.A., F.R.S.

the fundamental principal frequency, but the rapid development of high-speed machinery in large units has brought about the necessity for considering the higher modes of motion as well. In turbo-generators, for example, the large units are now constructed in such a manner that the second mode of oscillation of the rotor is of more vital importance than the fundamental, because it is more apt to conflict with the operating speed.

The following will describe a method for the expansion and the solution of the system determinant, which the author has used for a considerable time in his engineering activity, and which he has found useful on account of the symmetry of the expressions involved.

The method does not involve any new principles of fundamental character. It depends primarily upon the writing of the determinantal equation in a definite form from which a set of preliminary solutions, corresponding to a set of arbitrary modes of vibration, are obtained. The correct solutions are then centred around these preliminary solutions, the transformation being carried out by a graphical process. The fundamental properties of this particular form of the determinantal equation arise from the fact that the unknown occurs only in the principal diagonal, and as inverse squares of the frequencies.

The empirical formula proposed by Dunkerley, for the calculation of whirling speeds in shafts *, represents an approximation of the relations set forth below. A considerable portion of the literature on the subject has been devoted to discussions of the validity and the degree of approximation of the Dunkerley formula. The present treatment has been suggested, to a considerable extent, by the relations employed by these authors †.

II. Establishing the Determinantal Equation.

In order to illustrate the methods by which the determinantal equation can be brought to the desired form, we shall consider the most important of the major engineering problems—namely, the determination of whirling speeds in

* Philosophical Transactions, vol. clxxxv. p. 279 (1894).

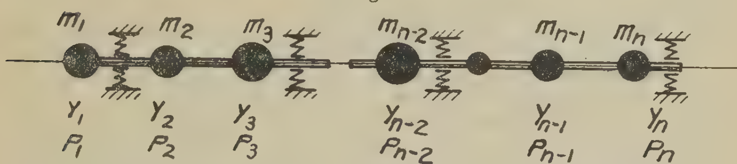
† E. Hahn, *Schweizerische Bauzeitung*, p. 191, Nov. 9, 1918: "Note sur la vitesse critique des arbres et le Formule de Dunkerley." Professor Hahn's treatment for whirling speeds differs but slightly from the present method. See also H. H. Jeffcott, Proc. Roy. Soc. vol. xcv. p. 106 (1918); R. C. Howland, Phil. Mag. vol. xlix. p. 1131 (1925); H. H. Jeffcott; Phil. Mag. p. 689 (April 1927).

multiple bearing arrangements, the determination of natural frequencies for torsional oscillations of line shafting with concentrated masses, and the determination of natural frequencies in mechanical systems of a more general type, such as are encountered in machine foundations, buildings, etc.

1. *Whirling Speeds in Shafts.*

Consider first the problem of whirling speeds in rotating shafts. The problem for shafts mounted in two bearings has been thoroughly investigated, and it is a well-known fact that the solution resolves itself into the determination of the natural frequencies of lateral vibrations. We shall consider the problem in a somewhat more general form, and refer to the system illustrated in fig. 1.

Fig. 1.



Lateral Vibrations of a System of Concentrated Masses on Elastic Shaft in Multiple Bearings.

Let there be n masses $m_1, m_2, \ldots, m_n$, which can be treated as distinct mass concentrations, interspersed with flexible, massless, stretches of shafting. This generalization may at first appear too extensive, but anyone familiar with the practical aspects of the problem will agree that there are very few of the systems encountered in practical problems that cannot be satisfactorily replaced by a system of concentrated masses on a weightless shaft. Bearing supports, flexibly connected to a rigid foundation reference, are introduced in an arbitrary order between these masses.

The action of the bearing supports depends upon the combined elasticity of the oil film and the bearing pedestals. In the ordinary problems of whirling motion it is usually permissible to neglect the masses of the bearing supports. However, there are no serious difficulties in taking them into account. The elasticity of the bearing pedestals may be considered as known, although it may be necessary to resort to static tests for its determination. The elasticity of the oil film presents the most uncertain part of this problem; and here it must be stated that further investigations are necessary before a fully reliable method for taking this

effect into account can be evolved. The available investigations on the subject * seem to indicate, however, that the oil film may, within certain limits, be considered as a spring of approximately constant flexibility. This conclusion is based on the fact that the motion of a rigid shaft in the oil film can be represented by linear differential equations.

In order to avoid excessive complications, it is assumed that the vibrations take place in one plane. Assuming a rotor structure of uniform flexibility in all directions, this is strictly permissible only when the flexibility of the bearing supports is the same for the vertical and the horizontal directions. This is usually not the case, because the flexibility of the bearing supports is frequently several times greater for horizontal motion than for vertical motion. However, the results obtained by treating the horizontal and vertical vibrations as independent motions seem to agree very well with experimental results.

Most engineering problems also permit the neglect of the gyroscopic forces; and where these terms are of importance, it is usually possible to apply a suitable correction.

Thus we may consider lateral vibrations of a system of masses, mechanically connected by a shaft of known elastic properties; this shaft is connected at certain points to a rigid foundation by mechanical members of known elasticity.

The differential equations of motion for this system are most easily established from the behaviour of the system under static loads $P_1, P_2, \dots P_n$, acting at the mass concentrations in the plane of motion. If $y_1, y_2, \dots, y_n$ represent the static displacements at the mass concentrations, the condition of static equilibrium appears in the following form:

$$\left.\begin{aligned} y_1 &= \alpha_{11}P_1 + \alpha_{12}P_2 + \dots\dots + \alpha_{1n}P_n \\ y_2 &= \alpha_{21}P_1 + \alpha_{22}P_2 + \dots\dots + \alpha_{2n}P_n \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ y_n &= \alpha_{n1}P_1 + \alpha_{n2}P_2 + \dots\dots + \alpha_{nn}P_n \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

The establishing of these equations is a problem of statics which never offers serious difficulties, although the expressions for the influence factors α may take complicated forms. Ordinarily the "moment method," where the bending moments at the supports are selected as statically indeterminate factors, leads most quickly to the result. For almost all engineering problems it is necessary to resort to

* Stodola, *Schweizerische Bauzeitung*, May 23, 1925. Hummel, *V.D.I. Forchungs Arbeiten*, Heft 287.

graphical constructions for determining the elastic properties of the individual spans.

Now, to obtain the differential equations of motion we change these static forces into inertia forces, and the static deflexions y into dynamic displacements—that is,

$$P_1 = -m_1\ddot{y}_1;\ P_2 = -m_2\ddot{y}_2;\ \ldots;\ P_n = -m_n\ddot{y}_n. \quad (2)$$

The determinantal equation, from which the principal frequencies are determined, is obtained by putting

$$y_1 = A_1 \sin \omega t;\ y_2 = A_2 \sin \omega t;\ \ldots;\ y_n = A_n \sin \omega t, \quad (3)$$

which, when introduced into the differential equations of motion, give the following condition for the independence of the integration constants A * :

$$\begin{vmatrix} 1-\alpha_{11}m_1\omega^2, & -\alpha_{12}m_2\omega^2, & \ldots, & -\alpha_{1n}m_n\omega^2, \\ -\alpha_{21}m_1\omega^2, & 1-\alpha_{22}m_2\omega^2, & \ldots, & -\alpha_{2n}m_n\omega^2, \\ \cdot & \cdot & \cdot & \cdot \\ -\alpha_{n1}m_1\omega^2, & -\alpha_{n2}m_2\omega^2, & \ldots, & 1-\alpha_{nn}m_n\omega^2. \end{vmatrix} = 0. \quad (4)$$

Now divide each column of this determinant by

$$-\alpha_{11}m_1\omega^2,\ -\alpha_{22}m_2\omega^2,\ \ldots,\ -\alpha_{nn}m_n\omega^2,$$

respectively, and put

$$X_1^2 = \frac{C^2}{\alpha_{11}m_1};\ X_2^2 = \frac{C^2}{\alpha_{22}m_2};\ \ldots;\ X_n^2 = \frac{C^2}{\alpha_{nn}m_n}, \quad (5)$$

$$\xi^2 = \frac{1}{C^2\omega^2}, \quad (6)$$

where C is an arbitrary constant. Then the determinant appears in the following form :

$$\begin{vmatrix} 1-\xi^2X_1^2, & \dfrac{\alpha_{12}}{\alpha_{22}}, & \ldots, & \dfrac{\alpha_{1n}}{\alpha_{nn}}, \\ \dfrac{\alpha_{21}}{\alpha_{11}}, & 1-\xi^2X_2^2, & \ldots, & \dfrac{\alpha_{2n}}{\alpha_{nn}}, \\ \cdot & \cdot & \cdot & \cdot \\ \dfrac{\alpha_{n1}}{\alpha_{11}}, & \dfrac{\alpha_{n2}}{\alpha_{22}}, & \ldots, & 1-\xi^2X_n^2. \end{vmatrix} = 0. \quad (7)$$

This is the form of the determinantal equation to which the simplified treatment will be applied.

* Howland, Phil. Mag. p. 513, March 1927, describes a method for obtaining the determinantal equation by means of the integral equation and graphical integration, which appears useful for many problems where the systems are not readily subdivided int distinct mass concentrations.

If the coupling terms were all zero, the quantities X_1, $X_2, \ldots, X_n$ would represent the principal frequencies of the system—that is, they would then be the n positive roots in $\frac{1}{\xi}$ of the equation (7). The units in which these frequencies are expressed depend upon the constant C ; $C=1$ expresses the quantities as circular frequencies, $C=\frac{1}{2\pi}$ as cycles per second, and $C=\frac{30}{\pi}$ as cycles per minute. There will be no ambiguity in retaining the symbol ω for the principal frequencies expressed in any of these units.

It has been found convenient to give these quantities a certain significance, even in those cases when the coupling terms are not zero. In the following they will be denoted as *component frequencies.* In many practical problems they represent rough approximations to the principal frequencies, in which cases they correspond to a set of quasi-principal modes of motion. However, the conception is in no way limited by the degree of approximation.

The chief function of these component frequencies is to furnish a convenient medium for expressing the principal frequencies. All the terms in equation (7) are non-dimensional, so that any convenient set of units may be used for expressing the component frequencies ; the roots in ω will then appear in the same units. The constants α are usually very small numbers and, for this reason, inconvenient to carry along in the expansion of the determinant. This objection is eliminated by the form (7) because it contains only ratios of quantities of the same order. Of equal importance is the fact that the determinant is expressed in terms of elastic constants, the masses appearing only through the component frequencies. The advantage of this point will appear in connexion with the expansion and solution of the determinant.

2. *Torsional Vibrations in Shafts.*

Consider next the problem of determining the natural frequencies in torsion of a shaft carrying $n+1$ mass concentrations (fig. 2). The positions of the bearings and their elastic properties do not enter into this problem.

We shall denote the rigidities of the shaft elements (in twisting moment per radian deflexion) by $k_1, k_2, \ldots, k_n$. The moments of inertia of the masses, with respect to the axis of the shaft, will be denoted by $m_1, m_2, \ldots, m_{n+1}$. Denoting by

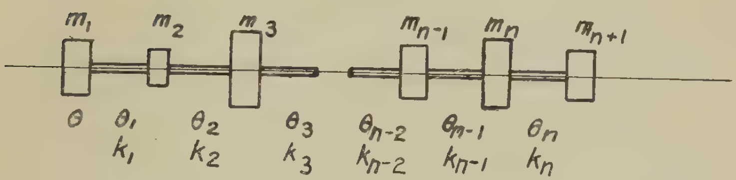
$\theta_1, \theta_2, \ldots, \theta_n$ the angles of twist in the shaft-sections between the masses, and by θ the angular displacement of m_1 with respect to a fixed direction of reference, we obtain for the kinetic energy

$$T=\tfrac{1}{2}m_1\dot{\theta}^2+\tfrac{1}{2}m_2(\dot{\theta}+\dot{\theta}_1)^2+\ldots\tfrac{1}{2}m_{n+1}(\dot{\theta}+\sum_{q=1}^{n+1}\dot{\theta}_q)^2, \quad (8)$$

and for the potential energy

$$V=\tfrac{1}{2}k_1\theta_1^2+\tfrac{1}{2}k_2\theta_2^2+\ldots+\tfrac{1}{2}k_n\theta_n^2. \quad . \quad . \quad (9)$$

Fig. 2.



Torsional Vibrations of a System of Concentrated Masses on Elastic Shaft.

The coordinate θ is ignorable, since it occurs only through the corresponding velocity. Thus we may obtain an integral of angular momentum :

$$\frac{\partial T}{\partial \dot{\theta}}=H, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where H is a constant, the known angular momentum of the entire system. There will be no loss of generality in putting $H=0$, because H will occur as a constant term in T and V. This simply implies that we calculate the natural frequencies for the system as a whole at rest.

Carrying out the ignoration of θ *, and establishing the Lagrangian equation of motion in the n remaining coordinates, we obtain

$$\left.\begin{array}{l} a_{11}\ddot{\theta}_1+a_{12}\ddot{\theta}_2+\ldots+a_{1n}\ddot{\theta}_n+k_1\theta_1=0 \\ a_{21}\ddot{\theta}_1+a_{22}\ddot{\theta}_2+\ldots+a_{2n}\ddot{\theta}_n+k_2\theta_2=0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_{n1}\ddot{\theta}_1+a_{n2}\ddot{\theta}_2+\ldots+a_{nn}\ddot{\theta}_n+k_n\theta_n=0 \end{array}\right\}, \quad . \quad (11)$$

* See, for example, Whittaker, 'Analytical Dynamics,' p. 53.

where the inertia constants a have the values :

$$a_{pp}=\frac{\sum\limits_1^p m \sum\limits_{p+1}^{n+1} m}{\sum\limits_1^{n+1} m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

$$a_{pq}=a_{qp}=\frac{\sum\limits_1^p m \sum\limits_{q+1}^{n+1} m}{\sum\limits_1^{n+1} m} \text{ when } p<q. \quad . \quad . \quad . \qquad (13)$$

It is evident from this result that the principal constants a_{pp} are made up from an imaginary system with *two* rigid masses : $\sum\limits_1^p m$ on the left of θ_p, and $\sum\limits_{p+1}^{n+1} m$ on the right of θ_p. The coupling constants a_{pq} are made up from an imaginary system with *three* rigid masses: $\sum\limits_1^p m$ on the left of θ_p, $\sum\limits_{p+1}^{q} m$ between θ_p and θ_q, and $\sum\limits_{q+1}^{n+1} m$ on the right of θ_q, only the outer elements entering into a_{pq}. The notations are such that a_{pq} is formed for $p<q$, and $a_{pq}=a_{qp}$. These conclusions are easily verified by developing the equations for a system with three masses. Thus the quasi-principal modes of vibrations are obtained by dividing the system into all possible combinations of three-mass systems *.

The determinantal equation for this system can now be written in the form :

$$\begin{vmatrix} k_1-a_{11}\omega^2, & -a_{12}\omega^2, & \ldots, & -a_{1n}\omega^2, \\ -a_{21}\omega^2, & k_2-a_{22}\omega^2, & \ldots, & -a_{2n}\omega^2, \\ \cdot & \cdot & \cdot & \cdot \\ -a_{n1}\omega^2, & -a_{n2}\omega^2, & \ldots, & k_n-a_{nn}\omega^2. \end{vmatrix} =0. \qquad (14)$$

By dividing each column by

$$-a_{11}\omega^2, \; -a_{22}\omega^2, \; \ldots, \; -a_{nn}\omega^2,$$

respectively, and putting

$$X_1{}^2=C^2\frac{k_1}{a_{11}}; \; X_2{}^2=C^2\frac{k_2}{a_{22}}; \; \ldots; \; X_n{}^2=C^2\frac{k_n}{a_{nn}} \quad .. \qquad (15)$$

and

$$\xi^2=\frac{1}{C^2\omega^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

* This representation of the general torsion problem has been used by Zerkowitz and Wydler. See Hans Wydler, 'Drehschwingungen in Kolbenmaschinenanlagen,' Julius Springer, 1922.

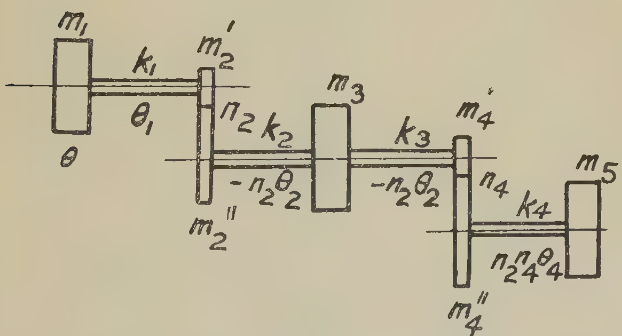
we now obtain

$$\begin{vmatrix} 1-\xi^2 X_1^2, & \frac{a_{12}}{a_{22}}, & \ldots, & \frac{a_{1n}}{a_{nn}}, \\ \frac{a_{21}}{a_{11}}, & 1-\xi^2 X_2^2, & \ldots, & \frac{a_{2n}}{a_{nn}}, \\ \cdot & \cdot & \cdot & \cdot \\ \frac{a_{n1}}{a_{11}}, & \frac{a_{n2}}{a_{22}}, & \ldots, & 1-\xi^2 X_n^2. \end{vmatrix} = 0. \quad (17)$$

This determinant is identical in form with (7), but the constants a now refer to the mass configuration, while the shaft constants k appear only through the component frequencies X.

An important variation of the torsion problem occurs in those cases where the different sections of the shaft are connected by gears. Fig. 3 is an example of such a system,

Fig. 3.



Torsional Vibrations of a System of Concentrated Masses on Geared Sections of Elastic Shaft.

where the mass m_2' is geared to m_2'' by a gear ratio n_2, and the mass m_4' is geared to m_4'' by a gear ratio n_4. It will be found that the treatment of systems of this type is identical with that of the simple system, by virtue of the fact that it is always possible to replace the geared system by a simple direct-connected system. The correspondence is best shown by evaluating the kinetic and potential energies for the

system in fig. 3. With the notations suggested in fig. 3 it is found that

$$\begin{aligned}T = &\tfrac{1}{2}m_1\dot{\theta}^2 + \tfrac{1}{2}(m_2' + n_2^2 m_2'')(\dot{\theta}+\dot{\theta}_1)^2 + \tfrac{1}{2}m_3 n_2^2(\dot{\theta}+\dot{\theta}_1+\dot{\theta}_2)^2 + \\ &+ \tfrac{1}{2}(m_4' + n_4^2 m_4'')n_2^2(\dot{\theta}+\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)^2 + \\ &+ \tfrac{1}{2}m_5 n_2^2 n_4^2(\dot{\theta}+\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3+\dot{\theta}_4)^2, \quad \ldots \ldots \quad (18)\end{aligned}$$

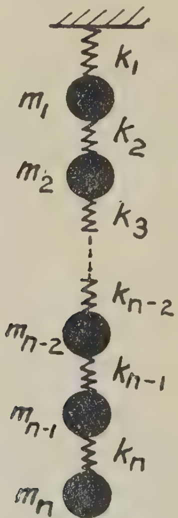
$$V = \tfrac{1}{2}k_1\theta_1^2 + \tfrac{1}{2}k_2 n_2^2\theta_2^2 + \tfrac{1}{2}k_3 n_2^2\theta_3^2 + \tfrac{1}{2}k_4 n_2^2 n_4^2\theta_4^2; \quad \ldots \quad (19)$$

from which expression it is evident that the determinantal equation will obtain the form (17). It is merely necessary to modify the inertia and the shaft constants in an appropriate manner. The most reliable method for this modification is obtained by writing the expressions for T and V before the determinant (17) is formed.

3. *Vibrations in other Configurations.*

Considering, finally, the general types of systems encountered in engineering problems, it is usually possible to apply either one of the treatments given above.

Fig. 4.



Example of Mechanical System.

Fig. 4 illustrates a frequently re-occurring system, where n masses are connected in series by springs. This system is merely a general case of that shown in fig. 2, and it is so

simple that it is hardly necessary to outline the steps which lead to the determinantal equation. It is worthy of note, however, that the determinantal equation for this system may be made to take the form (7), where all terms, outside of the principal diagonal, are expressed through the elastic constants of the springs. In order to obtain this form, it is merely necessary to select as coordinates the absolute displacements of the masses, and either apply the method given for fig. 1 or write the Lagrangian equations in the usual manner. It is found that the constants α take the values:

$$\alpha_{pp}=\sum_{1}^{p}\frac{1}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

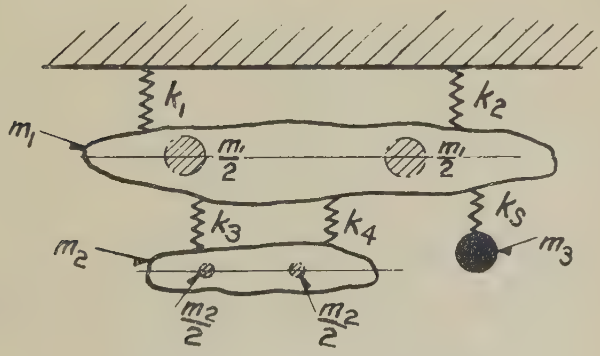
and
$$\alpha_{pq}=\alpha_{qp}=\sum_{1}^{p}\frac{1}{k} \text{ when } p<q. \quad . \quad . \quad . \quad (21)$$

The determinantal equation may also be written in such a manner that all terms, outside of the principal diagonal, are expressed through the masses (equation 17). In order to obtain this form, it is merely necessary to select as coordinates the relative displacements of the masses and write the Lagrangian equations of motion. The constants a take the values:

$$a_{pp}=\sum_{p}^{n}m \quad . \quad . \quad . \quad . \quad \quad . \quad . \quad . \quad (22)$$

and
$$a_{pq}=a_{qp}=\sum_{p}^{n}m \text{ when } p<q. \quad . \quad . \quad . \quad (23)$$

Fig. 5.



Example of Mechanical System.

Fig. 5 illustrates another type of system which frequently occurs in studies of machine foundations. The masses m_1

and m_2 are assumed to vibrate with small amplitudes in the plane of the paper, the possible displacements being along the directions of the springs, so that this particular system has five degrees of freedom. This type of system may also be made to yield a determinantal equation of either the form (7) or (17). In the former case the absolute displacements of the equivalent masses, and in the latter case the extensions of the springs, are taken as coordinates.

III. Solution of the Determinantal Equation.

It is seen, therefore, that the determinantal equation for any of those systems which we have considered, and which constitute the major portion of important engineering problems, may be written in the form

$$\begin{vmatrix} 1-\xi^2X_1^2, & \frac{a_{12}}{a_{22}}, & \ldots, & \frac{a_{1n}}{a_{nn}}, \\ \frac{a_{21}}{a_{11}}, & 1-\xi^2X_2^2, & \ldots, & \frac{a_{2n}}{a_{nn}}, \\ \cdot & \cdot & \cdot & \cdot \\ \frac{a_{n1}}{a_{11}}, & \frac{a_{n2}}{a_{22}}, & \ldots, & 1-\xi^2X_n^2, \end{vmatrix} = 0, \qquad (24)$$

where the constants a may be either elastic constants or inertia constants. ξ represents the inverse value of the principal frequencies to be determined, and the quantities X represent the component frequencies, obtained by considering certain obvious modes of oscillation. They may or may not constitute close approximations to the principal frequencies, and the process of the solution may be considered as a proces of converting the component frequencies X into principal frequencies.

Expanding the determinant in descending powers of ξ, we obtain

$$(-1)^n\Bigg[\xi^{2n}-\xi^{2n-2}\sum_{p=1}^{n}\frac{1}{X_p^2}+\xi^{2n-4}\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{D_{pq}}{X_p^2X_q^2}-$$

$$-\xi^{2n-6}\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{n}\frac{D_{pqr}}{X_p^2X_q^2X_r^2}+\ldots-\frac{D}{X_1^2X_2^2\ldots X_n^2}\Bigg]=0, \quad (25)$$

where the summations are extended over the possible permutations of n terms, each permutation taken once only.

The quantities D, D_{pq}, D_{pqr}, etc., represent the determinant

$$D=\begin{vmatrix} 1, & \frac{a_{12}}{a_{22}}, & \dots, & \frac{a_{1n}}{a_{nn}}, \\ \frac{a_2}{a_{11}}, & 1, & \dots, & \frac{a_{2n}}{a_{nn}}, \\ \cdot & \cdot & \cdot & \cdot \\ \frac{a_{n1}}{a_{11}}, & \frac{a_{n2}}{a_{22}}, & \dots, & 1, \end{vmatrix} \quad . \quad . \quad . \quad . \quad (26)$$

and its minors of the principal diagonal terms, thus:

$$D_{pq}=\begin{vmatrix} 1, & \frac{a_{pq}}{a_{qq}}, \\ \frac{a_{qp}}{a_{pp}}, & 1, \end{vmatrix}; \quad D_{pqr}=\begin{vmatrix} 1, & \frac{a_{pq}}{a_{qq}}, & \frac{a_{pr}}{a_{rr}}, \\ \frac{a_{qp}}{a_{pp}}, & 1, & \frac{a_{qr}}{a_{rr}}, \\ \frac{a_{rp}}{a_{pp}}, & \frac{a_{rq}}{a_{qq}}, & 1, \end{vmatrix}, \quad (27)$$

and so on. It should be noted that the indices for the minors refer to the terms actually contained in the determinants, and not, as is customary, to the diagonal terms to which they correspond. This notation has been selected to obtain symmetry of the terms in equation (25).

Equation (25) has n positive roots in ξ^2, corresponding to n principal modes of vibration. From the general properties of algebraic equations of this type, we can establish the following n relations between the principal frequencies $\omega_1, \omega_2, \dots, \omega_n$ and the known component frequencies $X_1, X_2, \dots, X_n$:

$$\left.\begin{aligned} \sum_{p=1}^{n}\frac{1}{\omega_p^{\,2}} &= \sum_{p=1}^{n}\frac{1}{X_p^{\,2}}=\frac{1}{S_1^{\,2}}, \\ \sum_{p=1}^{n}\sum_{q=1}^{n}\frac{1}{\omega_p^{\,2}\omega_q^{\,2}} &= \sum_{p=1}^{n}\sum_{q=1}^{n}\frac{D_{pq}}{X_p^{\,2}X_q^{\,2}}=\frac{1}{S_2^{\,2}}, \\ \cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot \quad \cdot \quad \cdot \\ \frac{1}{\omega_1^{\,2}\omega_2^{\,2}\dots\omega_n^{\,2}} &= \frac{D}{X_1^{\,2}X_2^{\,2}\dots X_3^{\,2}}=\frac{1}{S_n^{\,2}}. \end{aligned}\right\} \quad . \quad . \quad (28)$$

The principal frequencies may thus be obtained as the roots of the algebraical equation

$$\xi^{2n}-\frac{1}{S_1^{\,2}}\xi^{2n-2}+\dots-\frac{1}{S_n^{\,2}}=0. \quad . \quad . \quad (29)$$

It is evident that this method of expanding the determinant

is useful, even when the roots are to be evaluated by solving equation (29). The form of the constants S is so uniformly symmetrical that there are no difficulties in evaluating them directly from the determinant, nor, in many cases, from the expressions for the static equilibrium, or the kinetic and potential energy.

However, the main advantage of this method of expansion lies in the fact that it is possible to derive a simple graphical method for constructing the roots from the quantities S. This method will be explained successively for two, three, and several degrees of freedom.

For systems with two degrees of freedom we have

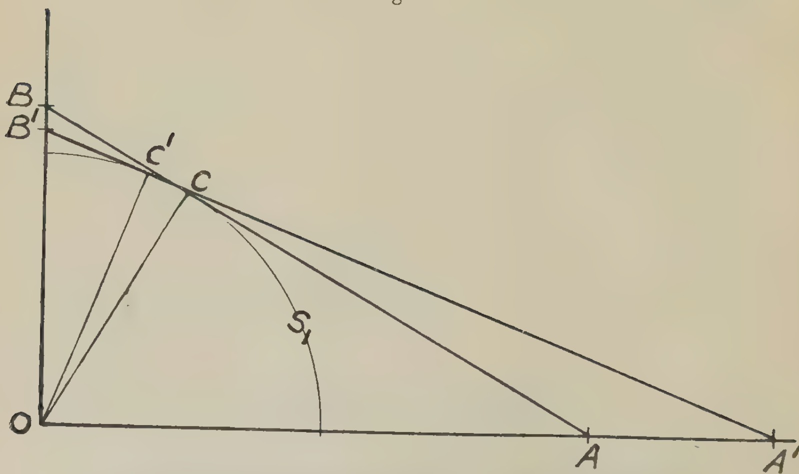
$$\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}=\frac{1}{X_1^2}+\frac{1}{X_2^2}=\frac{1}{S_1^2}, \quad . \quad . \quad . \quad (30)$$

$$\frac{1}{\omega_1^2\omega_2^2}=\frac{D}{X_1^2X_2^2}=\frac{1}{S_2^2}, \quad . \quad . \quad . \quad . \quad (31)$$

where

$$D=\begin{vmatrix} 1, & \frac{a_{12}}{a_{22}}, \\ \frac{a_{21}}{a_{11}}, & 1, \end{vmatrix}. \quad . \quad . \quad . \quad . \quad (32)$$

Fig. 6.



Converting Component Frequencies into Principal Frequencies for Systems with Two Degrees of Freedom.

Now, if $X_1=OA$ and $X_2=OB$ are set off along the axes of a rectangular system of coordinates (fig. 6), we find immediately, from equation (30), that the perpendicular OC

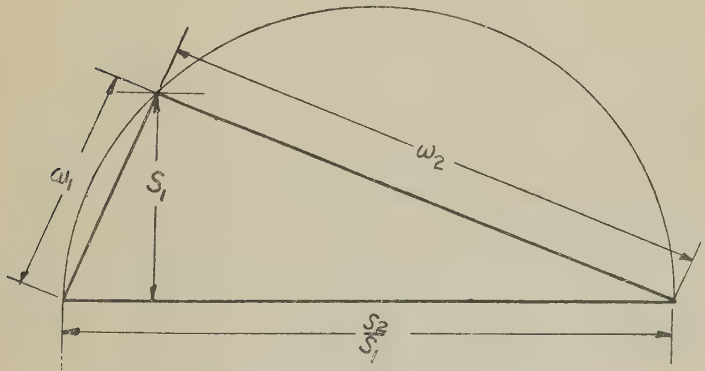
to the diagonal AB is equal to S_1. It is evident that the diagonal A′B′, which corresponds to the principal frequencies $\omega_1 = OA'$ and $\omega_2 = OB'$, lies at the same distance from O, so that it is a tangent to the circle S_1. Consequently, the component frequencies are converted into principal frequencies by rolling the diagonal AB on the circle S_1 until its position satisfies equation (31), when its length is increased to $A'B' = \frac{AB}{\sqrt{D}}$. This gives a satisfactory conception of the manner in which the component frequencies differ from the principal frequencies. If, for example, X_2 is much smaller than X_1, it will change very slightly, even if the coupling factor $\sqrt{D}$ is far from unity, while X_1 will change considerably. In all cases the lowest of the component frequencies is lowered while the upper is raised.

For practical purposes it is more convenient to use a somewhat modified form of the construction. By dividing equation (31) into (30), we obtain

$$\omega_1^2 + \omega_2^2 = \frac{S_2^2}{S_1^2} \quad \ldots \ldots \quad (33)$$

Thus, if we draw a semicircle on the base line $\frac{S_2}{S_1}$ (fig. 7),

Fig. 7.



Graphical Construction of the Principal Frequencies for Two Degrees of Freedom.

and intersect this semicircle by a line parallel to the diameter at a distance S_1, the latter will give a point which determines the correct values of ω_1 and ω_2.

In spite of the fact that the solution for two degrees of freedom is easily obtained by solving a quadratic equation,

it has been found worth while to use this graphical method, particularly in cases where the solution must be obtained a great number of times for different modifications of the same system.

Considering next a case with three degrees of freedom, we have

$$\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}+\frac{1}{\omega_3^2}=\frac{1}{X_1^2}+\frac{1}{X_2^2}+\frac{1}{X_3^2}=\frac{1}{S_1^2}, \quad . \quad (34)$$

$$\frac{1}{\omega_1^2\omega_2^2}+\frac{1}{\omega_1^2\omega_3^2}+\frac{1}{\omega_2^2\omega_3^2}=\frac{D_{12}}{X_1^2X_2^2}+\frac{D_{13}}{X_1^2X_3^2}+\frac{D_{23}}{X_2^2X_3^2}=\frac{1}{S_2^2}, \quad . \quad . \quad . \quad . \quad (35)$$

$$\frac{1}{\omega_1^2\omega_2^2\omega_3^2}=\frac{D}{X_1^2X_2^2X_3^2}=\frac{1}{S_3^2}, \quad . \quad . \quad . \quad (36)$$

where

$$\left.\begin{aligned}
D&=\begin{vmatrix} 1, & \frac{a_{12}}{a_{22}}, & \frac{a_{13}}{a_{33}}, \\ \frac{a_{21}}{a_{11}}, & 1, & \frac{a_{23}}{a_{33}}, \\ \frac{a_{31}}{a_{11}}, & \frac{a_{32}}{a_{22}}, & 1, \end{vmatrix}; \quad D_{12}=\begin{vmatrix} 1, & \frac{a_{12}}{a_{22}}, \\ \frac{a_{21}}{a_{11}}, & 1, \end{vmatrix}; \\
D_{13}&=\begin{vmatrix} 1, & \frac{a_{13}}{a_{33}}, \\ \frac{a_{31}}{a_{11}}, & 1, \end{vmatrix}; \quad D_{23}=\begin{vmatrix} 1, & \frac{a_{23}}{a_{33}}, \\ \frac{a_{32}}{a_{22}}, & 1, \end{vmatrix}.
\end{aligned}\right\} \quad . \quad (37)$$

Using a representation similar to the one shown on fig. 6 for a three-dimensional system of coordinates, it is evident that the three component frequencies are converted into principal frequencies by rolling the diagonal plane ABC on the sphere S_1 until it is changed into the plane A′B′C′, satisfying equations (35) and (36).

The practical solution by graphical construction corresponds to the one given in fig. 7, and is carried out in the following manner.

Dividing equation (36) into (35), we obtain

$$\omega_1^2+\omega_2^2+\omega_3^2=\frac{S_3^2}{S_2^2}. \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Substituting from equations (34) and (36) in (35), the latter may be written in the following form :

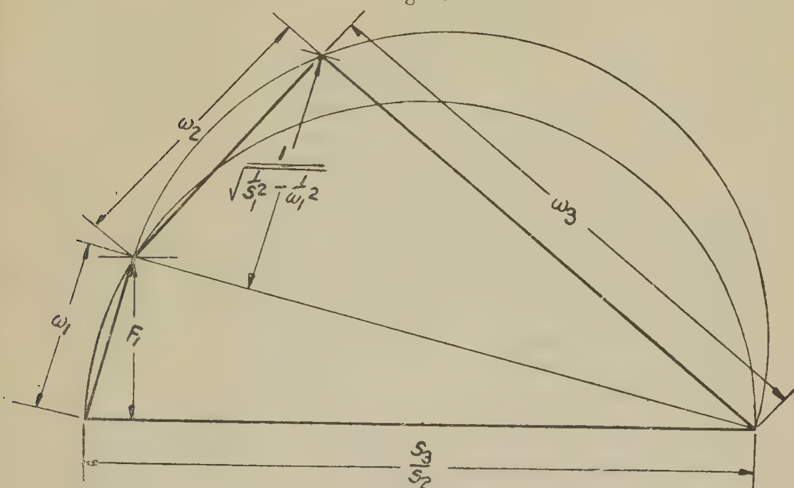
$$\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2+\omega_3^2}=\frac{1}{S_2^2\left[\frac{1}{S_1^2}-\frac{1}{\omega_1^2}\right]}=\frac{1}{p_1^2}, \quad . \quad (39)$$

so that

$$p_1 = S_2\sqrt{\frac{1}{S_1^2} - \frac{1}{\omega_1^2}}. \quad . \quad . \quad . \quad (40)$$

Now, drawing a semicircle on the base line $\frac{S_3}{S_2}$ (fig. 8), we try several values of ω_1 (starting with a value slightly

Fig. 8.



Graphical Construction of the Principal Frequencies for Three Degrees of Freedom.

above S_1) in equation (40) until the graphical construction yields a value of ω_1 equal to the assumed value, when ω_1 is established. The remaining chord in the semicircle now has the length $\sqrt{\omega_2^2 + \omega_3^2}$, and the remaining frequencies are obtained by drawing a new semicircle over this distance and intersecting it at a distance equal to

$$\frac{1}{\sqrt{\frac{1}{S_1^2} - \frac{1}{\omega_1^2}}},$$

which has already been established in finding ω_1.

The construction for three degrees of freedom is thus characterized by one operation based on cut and trial, and a second on straightforward construction. Naturally, it is also possible to obtain all three roots by cut and trial, but this method possesses no marked advantage over the solution of the cubic.

No attention has been given so far to the order in which the roots are obtained. Naturally it is not always necessary that the roots appear in the order of magnitude as shown on fig. 8. This feature is of no consequence, as long as the result consists of a polygon $\omega_1-\omega_2-\omega_3$, closed by the base line S_3/S_2.

The process just given for three degrees of freedom establishes a pattern which can be followed for any number of roots. The number of cuts and trials is always two less than the number of roots.

In the general case, with n degrees of freedom, we have n equations, giving the relations between the inverse squares of the frequencies in terms of the quantities $S_1, S_2, \ldots, S_n$. The two last equations can be combined into

$$\sum_{p=1}^{n} \omega_p^2 = \frac{S_n^2}{S_{n-1}^2}, \quad \ldots \ldots \quad (41)$$

which establishes the base line S_n/S_{n-1}. Now, if we want to solve for a certain root ω_1, we obtain the corresponding perpendicular from the base line p_1 by rearranging the $(n-1)$th equation into

$$\frac{1}{\omega_1^2}+\frac{1}{\sum\limits_{p=2}^{n}\omega_p^2}$$

$$=\frac{1}{S_{n-1}^2\left\{\frac{1}{S_{n-2}^2}-\frac{1}{\omega_1^2}\left[\frac{1}{S_{n-3}^2}-\frac{1}{\omega_1^2}\left(\cdots-\frac{1}{\omega_1^2}\left(\frac{1}{S_1^2}-\frac{1}{\omega_1^2}\right)\right)\right]\right\}}; \quad \ldots \quad (42)$$

that is,

$$p_1=S_{n-1}\sqrt{\frac{1}{S_{n-2}^2}-\frac{1}{\omega_1^2}\left[\frac{1}{S_{n-3}^2}-\frac{1}{\omega_1^2}\left(\cdots-\frac{1}{\omega_1^2}\left(\frac{1}{S_1^2}-\frac{1}{\omega_1^2}\right)\right)\right]}. \quad \ldots \quad (43)$$

When the root ω_1 is found a new base line is obtained, and a new set of constants t, corresponding to S, are immediately given. It is evident that

$$\frac{1}{t_1^2}=\frac{1}{S_1^2}-\frac{1}{\omega_1^2}, \quad \ldots \ldots \quad (44)$$

$$\frac{1}{t_2^2}=\frac{1}{S_2^2}-\frac{1}{\omega_1^2}\left(\frac{1}{S_1^2}-\frac{1}{\omega_1^2}\right), \quad \ldots \ldots \quad (45)$$

. . . .

$$\frac{1}{t_{n-2}^2}=\frac{1}{S_{n-2}^2}-\frac{1}{\omega_1^2}\left[\frac{1}{S_{n-3}^2}-\frac{1}{\omega_1^2}\left(\cdots-\frac{1}{\omega_1^2}\left(\frac{1}{S_1^2}-\frac{1}{\omega_1^2}\right)\right)\right]. \quad (46)$$

These constants were all established in the process of finding ω_1.

The perpendicular to the new base line, which determines ω_2, is now given by

$$p_2 = t_{n-2}\sqrt{\frac{1}{t_{n-3}^2} - \frac{1}{\omega_2^2}\left[\frac{1}{t_{n-4}^2} - \frac{1}{\omega_2^2}\left(\ldots - \frac{1}{\omega_2^2}\left(\frac{1}{t_1^2} - \frac{1}{\omega_2^2}\right)\right)\right]}. \quad . \quad . \quad . \quad (47)$$

As soon as ω_2 is given, a new base line is drawn, and a new set of constants u are determined in the same manner. The process is repeated until only ω_{n-1} and ω_n remain; these are obtained directly by intersecting the corresponding semicircle by the perpendicular

$$p_{n,\,n-1} = \frac{1}{\sqrt{\frac{1}{S_1^2} - \sum_{p=1}^{n-3}\frac{1}{\omega_p^2}}}. \quad . \quad . \quad . \quad (48)$$

Certain general conclusions as to the approximate magnitude of the roots are obtained from the semi-principal frequencies and from the equations (28). It is evident from the first equation, which is nothing but the usual Dunkerley formula, that no root can be smaller than S_1; in the case of a large number of roots, S_1 will have a value very near to the lowest. Similarly, there can be no root greater than S_n/S_{n-1}, but the largest root is usually considerably below this value.

The question of accuracy of the result is of paramount importance. It is evident that the construction suffers from accumulation of errors for the higher roots. The author has found that, with reasonably careful drawing work on letter-size paper, the accuracy is quite sufficient for practical purposes, at least up to five or six roots. It is a general feature of most practical problems that only a few of the lower roots are of practical interest, so that it is not believed that the lack of accuracy will prohibit the use of the method for ordinary analysis of practical problems.

The major part of the arithmetical work is involved in the evaluation of the determinants and the subsequent calculation of the factors S. One of the most important advantages of the method lies in the circumstance that the determinants may usually be made to express either the elastic properties or the inertia properties of the system. In most engineering problems it is a question of obtaining a series of solutions for different magnitudes of the various elements. Sometimes it is required to find a suitable set of elastic properties for a given set of inertia, whilst at other times (perhaps more rarely) it may be desired to find a suitable set of masses for a given set of elastic constants. In such cases, and they constitute the most important and

most fruitful aspect of vibration analysis, it is possible to arrange the analysis so that the evaluation of the determinants is carried out once only, while the adjustment of the system is expressed by variations in the component frequencies.

IV. Numerical Examples.

Figs. 9 and 10 have been given as examples of calculations for two representative cases.

Fig. 9 is a calculation of the whirling speeds of a turbine generator mounted in three bearings. The calculation has been carried out both for the case of perfectly rigid bearings and for the case of a practical set of values for the bearing flexibilities. In this problem it is possible to replace the system by three concentrated masses. The turbine spindle is comparatively short, so that the frequencies corresponding to its second mode of motion are very high. For this reason it is sufficiently accurate to represent it as a single mass, m_3. The generator rotor, on the other hand, is comparatively long, and here it is necessary to consider its second mode of motion. As a matter of interest, this particular rotor was analysed because the whirling speed corresponding to the second mode of motion of the generator rotor comes quite close to the operating speed (3600 R.P.M.).

The bearing flexibilities δ have values of the same order as those found on similar installations, and they represent bearing conditions which can be approximated only by a very robust design.

The influence factors α have been determined by graphical analysis of the actual rotor shaft. The positions of the equivalent masses were found by computing the radius of gyration of the middle body *. It is of interest to note that for rotors of this type the treatment suggested by Rodgers gives results in close agreement with more rigorous methods for the determination of the second whirling speed, as given by Stodola †.

The calculation is self-explanatory. It is evident from the results that the bearing flexibility assumed does lower the fundamental critical speed an appreciable amount.

Fig. 10 is an example of a calculation of critical frequencies for torsion in a system of five masses. This example was selected from a Diesel electric ship drive.

These diagrams were drawn on base lines of about 6 inches in length, and the accuracy is, in all cases, well within 1 per cent. of the true values obtained by arithmetical computation.

* See Rodgers, Phil. Mag. p. 122, July 1922.

† Stodola, ' Dampf und Gasturbinen,' p. 386.

V. *On the Emission of Positive Electricity from Hot Tungsten in Mullard Radio Valves.* By PHANINDRA KUMAR MITRA, *M.Sc.**

IN a recent paper † Prof. W. A. Jenkins has shown that there is a copious emission of positive electricity from a hot tungsten wire which forms the anticathode of a Coolidge Tube at a very high temperature (beginning from about 2000° C. to 3500° C., *i.e.*, up to the melting-point of tungsten). The present investigator worked on the same lines, but with Mullard radio valves instead of with Coolidge tubes, partly because of the high price of Coolidge tubes and partly because with Mullard radio valves different types of cathodes of various sizes could be used with advantage. The present paper is an account of the variation of positive emission (i.) with temperature, (ii.) with applied potential difference; and (iii.) of growth and (iv.) decay of positive emission current with time under various conditions.

The arrangement of apparatus is, however, different from that used by Prof. Jenkins. In his arrangement he balanced the difference of potentials between the ends of a hot tungsten wire against the E.M.F. of a 2-volt storage cell, and knowing the current flowing through the tungsten wire, calculated its resistances when different currents were flowing through it. But in the present case an arrangement exactly similar to that used by Richardson ‡ has been made. The hot wire forms the fourth arm of a Wheatstone's bridge, two resistance boxes being made the first two arms, while a wire of low temperature-coefficient placed in a constant-temperature oil-bath forms the third arm of the bridge. The arrangement is shown below.

P, Q are two resistance boxes from which resistances of the order of 10,000$^{\omega}$ are generally unplugged. R is a non-inductive wire resistance of low temperature-coefficient placed in an oil-bath at constant temperature. T is a sensitive thermometer which shows any variation in temperature of the bath. The bath is placed on a vessel V provided with water-cooling arrangement. S is the tungsten anode of a Mullard radio valve, while C is made the cathode. G_1 is a galvanometer of the order of 10^{-8} ampere per 1 mm. division. X is a set of batteries of 12 2-volt accumulators. It supplies the heating current, as well as the current necessary for Wheatstone's net arrangement. P, Q being moderately

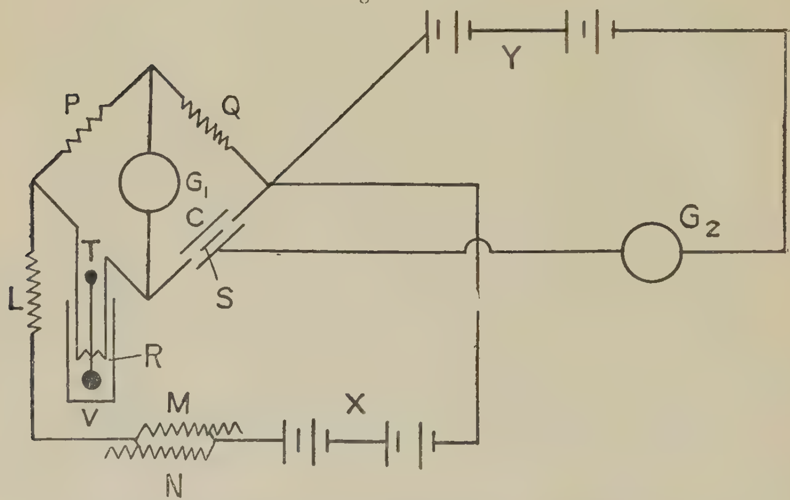
* Communicated by Prof. S. N. Bose, M.Sc.

† Phil. Mag. (6) xlvii. p. 1025.

‡ Richardson, 'Emission of Electricity from Hot Bodies,' p. 16.

high resistances, the major part of the current passes through R and S. L is a resistance in the battery circuit for adjustment of current; while M, N, two resistances joined in parallel, are meant for fine adjustment of current. G_2 is a very high-sensitivity galvanometer of the order of 10^{-10} ampere per 1 mm. division used to measure the positive emission current. Y is a set of Exide batteries from which voltage from 2 to 200 could be easily and conveniently applied. The positive terminal of Exide batteries is joined with the anode, while the negative one is joined through the high-sensitivity galvanometer with the cathode of the radio valve.

Fig. 1.



In working with this high-sensitivity galvanometer, very great difficulty was experienced in eliminating the leak. In moist weather, even without the application of any voltage at all, a deflexion too great to remain on the scale was sometimes observed. To avoid leakage, the two sets of batteries had to be placed on two different tables, which in their turn were made to stand on ebonite blocks. The working tables as well were insulated by standing them on ebonite blocks. Every individual piece of apparatus was placed on blocks of paraffin wax for the purpose of insulation. Even with these precautions inclement weather affected the arrangement so much that carrying out experiments was out of the question.

I. *Variation of Positive Emission with Temperature.*

Keeping the applied voltage constant, the heating current was gradually increased in steps, the positive emission current in each case was determined directly by the deflexion of the high-sensitivity galvanometer, while the balancing-point in the Wheatstone's net was obtained at each successive step by the low-sensitivity galvanometer. Thus the resistances of the tungsten filament could be determined as different currents flowed through it. Its initial resistance being known, temperatures of the filament could be calculated with the help of Langmuir's data *. Experiments were carried out with valves U 30 Nos. 311 and 312, as well as with U 150 No. 72 and U 250 No. 15. With the first two tubes the initial resistances of the filament could be easily determined, while with the last two the initial resistances could not be accurately known, as the resistances of leads were appreciable and could not be determined without breaking the tubes altogether. Table I. shows the data for the first two tubes. The initial resistance of the first tube was 0·1480 ohm, while that of the second was 0·1481 ohm.

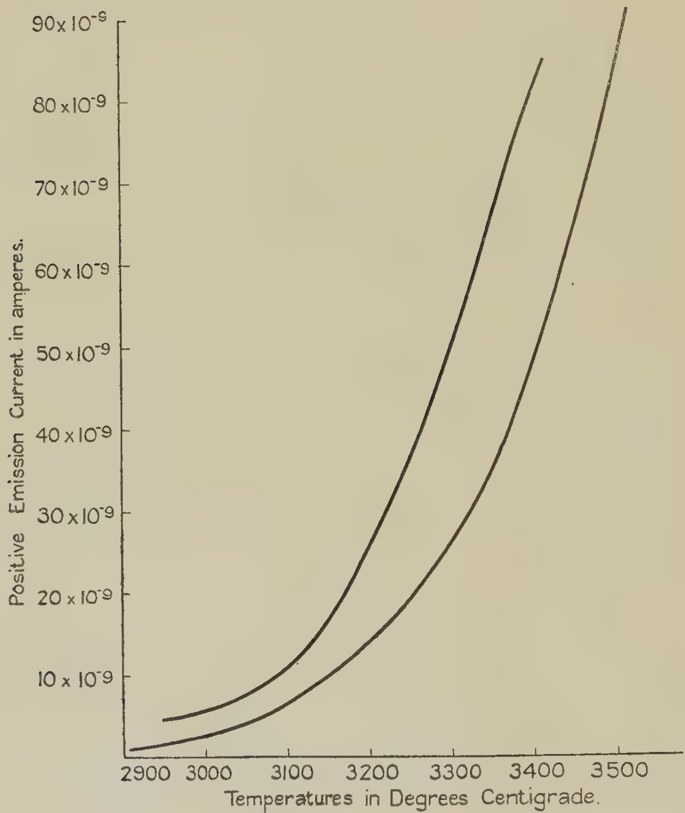
TABLE I.

Tube U 30 No. 311.			Tube U 30 No. 312.		
Resistance of Filament.	Temperature of Filament.	Positive Emission, in amperes.	Resistance of Filament.	Temperature of Filament.	Positive Emission, in amperes.
2·593$^{\omega}$	2946° C.	$4{\cdot}5\times10^{-9}$	2·552$^{\omega}$	2908° C.	1×10^{-9}
2·687	3033	7·5 „	2·651	2984	2·5 „
2·768	3108	10 „	2·711	3054	4 „
2·813	3152	16 „	2·834	3150	9·75 „
2·841	3180	21 „	2·949	3280	22·75 „
2·872	3207	27·5 „	3·004	3334	32·25 „
2·899	3235	34 „	3·053	3382	40 ,
2·942	3277	43·5 „	3·094	3420	54 „
2·967	3300	49 „	3·146	3472	71·5 „
3·010	3343	63 „	3·194	3517	92·5 „
3·089	3414	85 „	—	—	—

From fig. 2 it will be seen that emissions from each tube are very nearly similar, and that they increase very rapidly with temperature, more rapidly than linear relation would indicate.

* Phys. Rev. (2) vii. p. 315.

Fig. 2.



II. *Variation of Positive Emission with Voltage.*

Keeping the heating current constant with the help of adjustable resistances, the applied voltage was successively increased from 2 to 100. With the valve tube U 150 No. 72, an applied voltage below 14 was not sufficient to completely resist the electron current. Even with 14 volts an electron current of $7 \cdot 5 \times 10^{-9}$ ampere was obtained. With the application of 16 volts and upwards there was a gradual rise of positive emission current. With tube U 250 No. 15 an applied E.M.F. below 12 volts was not found sufficient to resist the electron current, and even with 12 volts there was an electron current of 46×10^{-9} ampere. With the application of 14 volts and upwards there was a gradual increase of positive emission current. Table II. shows the data giving

Table II.

Tube U 150 No. 72.

Potential applied, in volts	14	16	18	20	22	24	26	28	30	32	34	36
Positive emission, in amperes......	$-7 \cdot 5 \times 10^{-9}$	$2 \cdot 5 \times 10^{-9}$	$7 \cdot 5 \times 10^{-9}$	$17 \cdot 5 \times 10^{-9}$	25×10^{-9}	$27 \cdot 5 \times 10^{-9}$	$28 \cdot 7 \times 10^{-9}$	35×10^{-9}	$42 \cdot 5 \times 10^{-9}$	$47 \cdot 5 \times 10^{-9}$	50×10^{-9}	51×10^{-9}

Potential applied, in volts	38	40	42	44	48	50	60	70	80	90	100
Positive emission, in amperes......	$52 \cdot 5 \times 10^{-9}$	55×10^{-9}	60×10^{-9}	60×10^{-9}	60×10^{-9}	60×10^{-9}	60×10^{-9}	60×10^{-9}	60×10^{-9}	60×10^{-9}	60×10^{-9}

Tube U 250 No. 15.

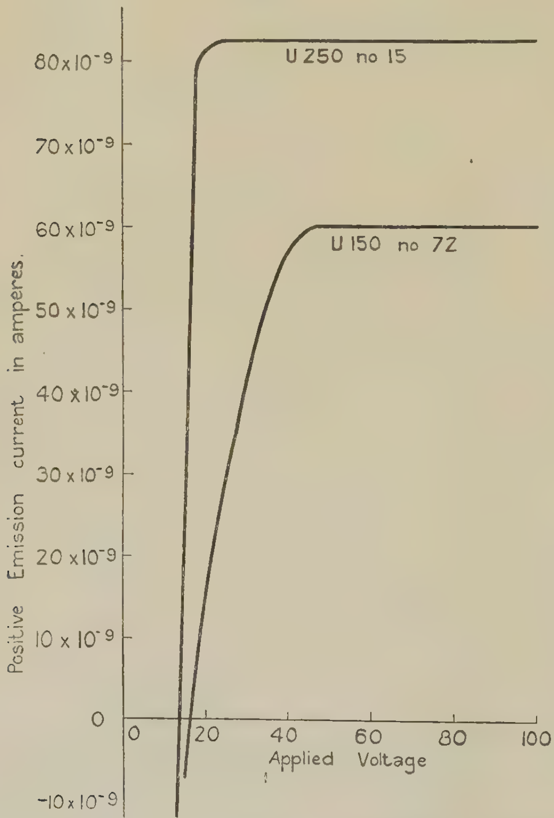
Potential difference applied, in volts	12	14	16	18	20	22	24	26	28
Positive emission current, in amperes	-46×10^{-9}	$7 \cdot 5 \times 10^{-9}$	$7 \cdot 75 \times 10^{-9}$	80×10^{-9}	$81 \cdot 25 \times 10^{-9}$	$83 \cdot 5 \times 10^{-9}$	$82 \cdot 5 \times 10^{-9}$	$82 \cdot 5 \times 10^{-9}$	$82 \cdot 5 \times 10^{-9}$

Potential difference applied, in volts	30	34	38	42	46	50	60	80	100
Positive emission current, in amperes	$82 \cdot 5 \times 10^{-9}$	$82 \cdot 5 \times 10^{-9}$	$82 \cdot 5 \times 10^{-9}$	$82 \cdot 5 \times 10^{-9}$	$82 \cdot 5 \times 10^{-9}$	$82 \cdot 5 \times 10^{-9}$	$82 \cdot 5 \times 10^{-9}$	$82 \cdot 5 \times 10^{-9}$	$82 \cdot 5 \times 10^{-9}$

the relation between the applied E.M.F. and the positive emission current for the tubes U 250 No. 15 and U 150 No. 72.

It will be readily seen from fig. 3 that a saturation value of the positive emission current is attained with a voltage of 42 with tube U 150 No. 72, while with tube U 250 No. 15

Fig. 3.



the saturation value of the current is attained with the application of 22 volts only. It should be noted, however, that with Coolidge tubes practically no saturation current was obtained.

III. *Growth of Positive and Negative Emission Currents with time.*

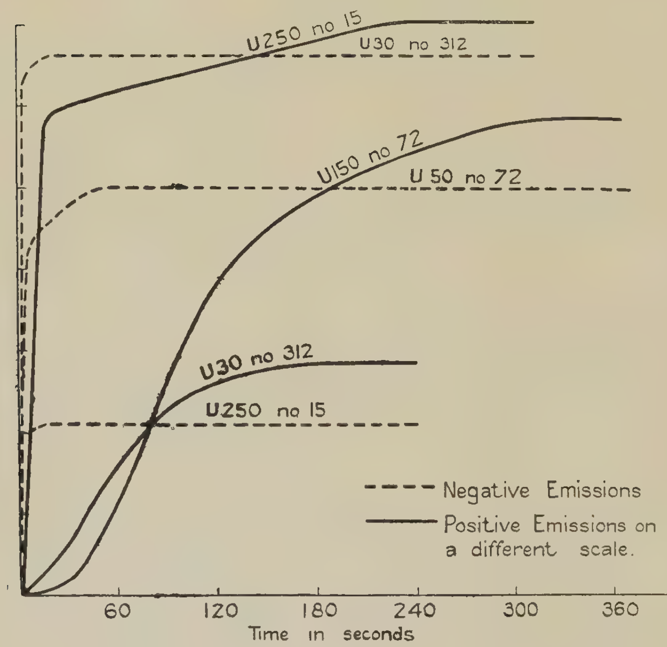
In these experiments, an E.M.F. sufficient to produce the saturation value of the positive emission current was applied

TABLE III.—Experiments with three valves: U 30 No. 312, U 250 No. 15, U 150 No. 72.

Time.	Tube U 30 No. 312. Potential applied, 36 volts.		Tube U 250 No. 15. Potential applied, 24 volts.		Tube U 150 No. 72. Potential applied, 54 volts.	
	Positive Emission Current, in amperes.	Negative Emission Current, in amperes.	Positive Emission Current, in amperes.	Negative Emission Current, in amperes.	Positive Emission Current, in amperes.	Negative Emission Current, in amperes.
5 sec.	$2\cdot5\times10^{-10}$	64×10^{-5}	0	20×10^{-5}	0	42×10^{-5}
15 sec.	7·5 ,,	66 ,,	$177\cdot5\times10^{-10}$	21 ,,	$2\cdot5\times10^{-10}$	45 ,,
30 sec.	20 ,,	66 ,,	180 ,,	21 ,,	5 ,,	48 ,,
45 sec.	33 ,,	66 ,,	182·5 ,,	21 ,,	20 ,,	50 ,,
1 min.	47·5 ,,	66 ,,	185 ,,	21 ,,	35 ,,	50 ,,
1 min. 30 sec.	70 ,,	66 ,,	190 ,,	21 ,,	80 ,,	50 ,,
2 min.	77·5 ,,	66 ,,	195 ,,	21 ,,	115 ,,	50 ,,
2 min. 30 sec.	82·5 ,,	66 ,,	200 ,,	21 ,,	135 ,,	50 ,,
3 min.	85 ,,	66 ,,	202·5 ,,	21 ,,	145 ,,	50 ,,
3 min. 30 sec.	85 ,,	66 ,,	207·5 ,,	21 ,,	155 ,,	50 ,,
4 min.	85 ,,	66 ,,	210 ,,	21 ,,	160 ,,	50 ,,
5 min.	—	—	210 ,,	21 ,,	170 ,,	50 ,,
6 min.	—	—	210 ,,	21 ,,	175 ,,	50 ,,
7 min.	—	—	—	—	175 ,,	50 ,,

at first and then the heating current was switched on and kept constant. Table III. and fig. 4 show the relation between the positive or negative emission current with time. It will be seen from fig. 4 that while the negative emission current attains its full value in less than half a minute, it requires from 3 to 6 minutes for the positive emission current to reach its maximum value.

Fig. 4.



IV. *Decay of Positive Emission Current with time.*

The applied potential difference was maintained constant, and was sufficient to produce saturation. The heating current was also kept constant. Tables IV., V., and VI. contain the experimental data showing the decay of positive emission current with time. Figs. 5, 6, and 7 show the characteristic curves for the decay of positive emission current. It will be noticed that the precise form of the current-time curves varies from one specimen of the wire to another. It depends on the previous treatment of the wires as well. The rate of decay is greater at a high temperature than at a low one. The current decays rapidly at first and

Table IV.

Type: U 500.
Potential difference applied, 64 volts.

Number of Tube. / Time, in minutes.	32. Positive Emission Current, in amperes.	37. Positive Emission Current, in amperes.	40. Positive Emission Current, in amperes.
0	$56\times5\times10^{-10}$	$125\times5\times10^{-10}$	$182\times5\times10^{-10}$
5	57 ,,	123 ,,	167 ,,
10	54 ,,	120 ,,	158 ,,
20	47 ,,	115 ,,	136 ,,
30	—	112 ,,	126 ,,
40	37·5 ,,	—	—
45	—	102 ,,	109 ,,
60	32·5 ,,	95 ,,	98 ,,
90	28 ,,	87 ,,	—
120	25 ,,	78·5 ,,	79 ,,
180	22 ,,	—	67·5 ,,
210	—	69 ,,	—
240	—	—	62·5 ,,

Table V.

Type of Valve: U 250.
Potential difference applied, 57 volts.

Number of Tube. / Time, in minutes.	15. Positive Emission, in amperes.	101. Positive Emission, in amperes.	102. Positive Emission, in amperes.	105. Positive Emission, in amperes.	108. Positive Emission, in amperes.
0	$129\times5\times10^{-10}$	$81{\cdot}5\times5\times10^{-10}$	$113\times5\times10^{-10}$	$113\times5\times10^{-10}$	$71{\cdot}5\times5\times10^{-10}$
5	—	79 ,,	113 ,,	108 ,,	56 ,,
10	—	74·5 ,,	108 ,,	101 ,,	50 ,,
15	133 ,,	—	—	—	—
20	—	67·5 ,,	98·5 ,,	90 ,,	41 ,,
30	121 ,,	62·5 ,,	90·5 ,,	82 ,,	30 ,,
45	113 ,,	55·5 ,,	83 ,,	75 ,,	24 ,,
60	103 ,,	49·5 ,,	76·5 ,,	69 ,,	21·5 ,,
90	90 ,,	40·5 ,,	67·5 ,,	59 ,,	19·5 ,,
120	79 ,,	34·5 ,,	59·5 ,,	49 ,,	15·5 ,,
150	—	29·5 ,,	—	—	13·5 ,,
180	64 ,,	26·5 ,,	43	34·5 ,,	12·5

TABLE VI.

Type of Tube: U 150.
Potential difference applied 67 volts.

Number of tube.	72.	128.	137.
Time, in minutes.	Positive Emission, in amperes.	Positive Emission, in amperes.	Positive Emission, in amperes.
0	$69 \times 5 \times 10^{-10}$	$83 \times 5 \times 10^{-10}$	$55{\cdot}5 \times 5 \times 10^{-10}$
5	69 ,,	82·5 ,,	53 ,,
10	68 ,,	80·5 ,,	50·5 ,,
20	64·5 ,,	77 ,,	46·5 ,,
30	61 ,,	75·5 ,,	43·5 ,,
45	57 ,,	73 ,,	40 ,,
60	54 ,,	71 ,,	37 ,,
90	48 ,,	69 ,,	33·5 ,,
120	44 ,,	68 ,,	30·5 ,,
150	40 ,,	—	—
188	38 ,,	65 ,,	27 ,,

TABLE VII.

Type of Valve: U 250 No. 110.
Applied potential difference 67 volts.

Time.	Positive Emission, in amperes.
0	$232 \times 5 \times 10^{-10}$
5 min.	225 ,,
10 min............	216 ,,
20 min............	189 ,,
30 min............	164 ,,
45 min............	143 ,,
1 hr....	127 ,,
1 hr. 30 min. ...	104 ,,
2 hrs.	90·5 ,,
3 hrs.	74 ,,
4 hrs.	61 ,,
5 hrs.	53 ,,
—	—
21 hrs.	29 ,,
24 hrs.	26 ,,
29 hrs.	23 ,,
—	—
46 hrs.	18 ,,
49 hrs.	17·5 ,,
54 hrs.	17 ,,

then more slowly, approaching a constant value i_0. It should be noted, however, that i_0 is not, strictly speaking, a constant.

Fig. 5.

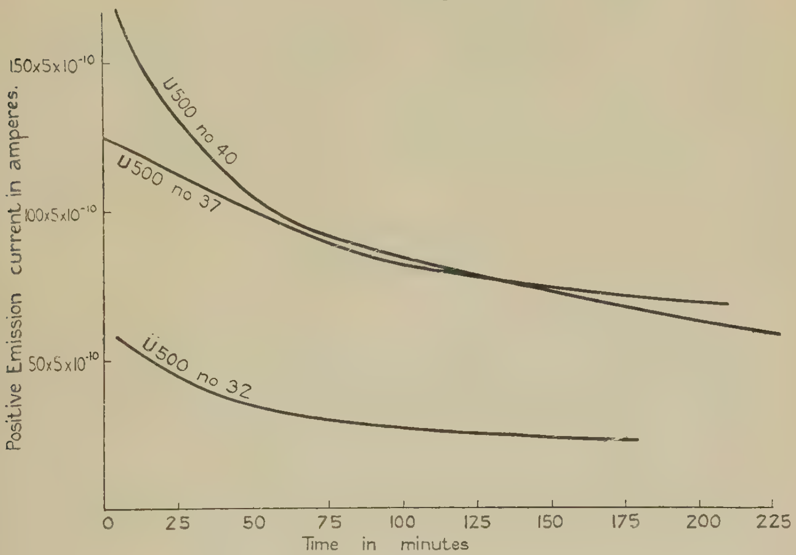
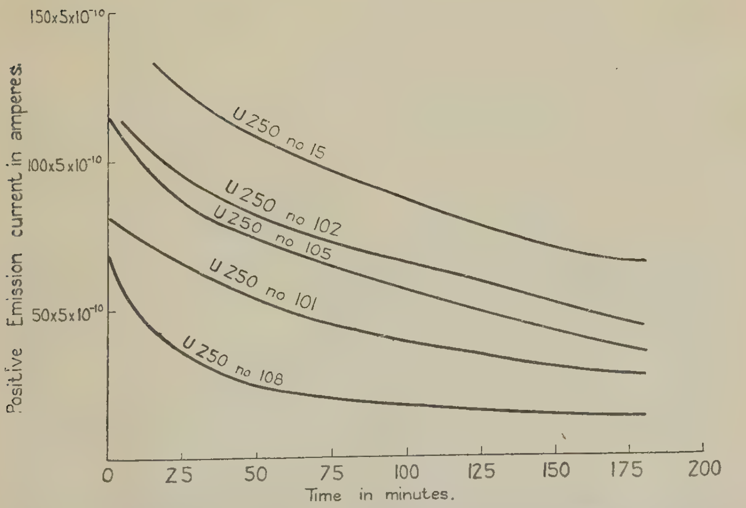


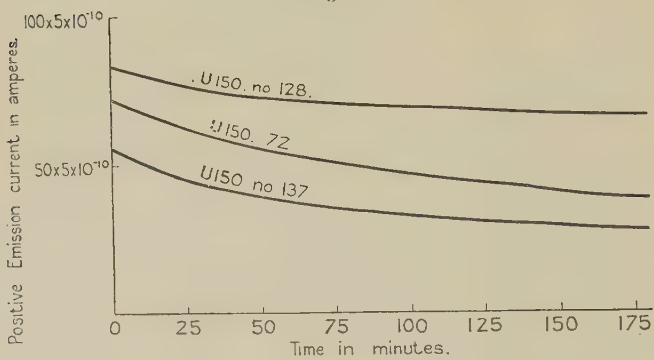
Fig. 6.



It also decays with time, but very slowly in the same general way. Table VII. shows the decay of positive emission

current with time with valve U 250, No. 110. It will be seen that a current goes down in the first stage from 232 to 53 in course of five hours, while in the second stage it goes down from 29 to 23 in eight hours, whereas in the final stage it goes down from 18 to 17, *i. e.*, only one division in course of eight hours. The decay in the first stage is very rapid, namely 179 in five hours, or about 36 divisions per hour. In the second stage it is about 0·75 division per hour; while in the last stage the decay is extremely slow, namely 0·12 division per hour.

Fig. 7.



As with a Coolidge tube, previous treatment of the valve plays a considerable part in the initial emission of positive electricity. It has been found that the application of the negative potential for about three hours with valve U 150 No. 72 increases the initial positive emission to twice its normal value. The experiment was made in the following way:—It was seen from the previous experiments that with the application of 54 volts there was a positive emission of about 32 divisions, and this positive emission decayed extremely slowly, so that we might reckon it as practically steady; now the applied potential was reversed, and this state of things continued for three hours; then, again, the potential was reversed. The initial positive emission current was now found to be about 60 divisions, but it decayed very rapidly. The following table shows this decay with time.

Tube U 150 No. 72. Potential applied, 54 volts.

Time.	Positive Emission Current, in amperes.
2 sec.	$60 \times 5 \times 10^{-10}$
5 sec.	40 ,,
15 sec.	35 ,,
30 sec.	34 ,,
45 sec.	33 ,,
1 min.	32 ,,
1 min. 30 sec.	32 ,,
2 min.	32 ,,
2 min. 30 sec.	32 ,,
3 min.	32 ,,

The theoretical aspects of this kind of positive emission current, which can be detected only at a very high temperature, will, it is hoped, be more fully discussed in a future paper.

VI. *A Note on Hamilton-Jacobi's Differential Equation in Dynamics. By* JAKOB KUNZ *.

THE fundamental importance of the differential equation of the title in the new quantum mechanics makes it desirable to obtain that equation in various, and if possible simple, ways. In the following note I wish to derive the differential equation of Hamilton-Jacobi directly from Euler's differential equation of the calculus of variation.

Let

$$f = f(x, y_1, y_2, \dots y_n, y_1', y_2' \dots y_n')$$

be a function of an independent variable x and n dependent variables $y_1, y_2 \dots y_n$, independent among themselves, and let

$$y_1' = \frac{dy_1}{dx}, \quad y_n' = \frac{dy_n}{dx}.$$

Then the integral

$$\mathrm{I} = \int f(x, y_1, y_2 \dots y_n, y_1', y_2' \dots y_n')\, dx$$

becomes an extremum if the following differential equations

* Communicated by the Author.

of Euler are satisfied :

$$\left.\begin{aligned} \frac{\partial f}{\partial y_1}-\frac{d}{dx}\left(\frac{\partial f}{\partial y_1'}\right)&=0 \\ \frac{\partial f}{\partial y_2}-\frac{d}{dx}\left(\frac{\partial f}{\partial y_2'}\right)&=0 \\ \vdots\qquad\qquad & \\ \frac{\partial f}{\partial y_n}-\frac{d}{dx}\left(\frac{\partial f}{\partial y_n'}\right)&=0 \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

In most of the classical examples of the calculus of variation it happens that f is explicitly independent of x, so that $\frac{\partial f}{\partial x}=0$; then the differential equations (1) can be put in another form, so that one integration can be carried out directly. Let us consider

$$\frac{d}{dx}\left(f-y_1'\frac{\partial f}{\partial y_1'}-y_2'\frac{\partial f}{\partial y_2'}\cdots-\frac{\partial f}{\partial y_n'}y_n'\right)$$

$$=\frac{\partial f}{dx}+\frac{\partial f}{dy_1}y_1'+\frac{\partial f}{\partial y_2}y_2'+\ldots+\frac{\partial f}{\partial y_n}y_n'$$

$$-y_1''\frac{\partial f}{\partial y_1'}-y_2''\frac{\partial f}{\partial y_2'}-\ldots-y_n''\frac{\partial f}{\partial y_1'}$$

$$-y_1'\frac{d}{dx}\frac{\partial f}{\partial y_1'}-y_2'\frac{d}{dx}\frac{\partial f}{\partial y_2'}-\ldots-y_n'\frac{d}{dx}\frac{\partial f}{\partial y_n'}$$

or

$$\frac{d}{dx}\left(f-y_1'\frac{\partial f}{\partial y_1'}-y_2'\frac{\partial f}{\partial y_2'}-\ldots-y_n'\frac{\partial}{\partial y_n'}\right)-\frac{\partial f}{dx}$$

$$=y_1'\left(\frac{\partial f}{\partial y_1}-\frac{d}{dx}\frac{\partial f}{\partial y_1'}\right)+y_2'\left(\frac{\partial f}{\partial y_2}-\frac{d}{dx}\frac{\partial f}{\partial y_2'}\right)+\ldots$$

$$+y^{n\prime}\left(\frac{\partial f}{\partial y_n}-\frac{d}{dx}\frac{\partial f}{\partial y_n'}\right). \quad . \quad (2)$$

If the equations (1) are satisfied, then the equation (2) reduces to

$$\frac{d}{dx}\left(f-y_1'\frac{\partial f}{\partial y_1'}-y_2'\frac{\partial f}{\partial y_2'}-\ldots-y_n'\frac{\partial f}{\partial y_n'}\right)-\frac{\partial f}{\partial x}=0. \quad (3)$$

and if $\frac{\partial f}{\partial x}=0$, then the first integration gives

$$f-y_1'\frac{\partial f}{\partial y_1'}-y_2'\frac{\partial f}{\partial y_2'}\dots-y_n'\frac{\partial f}{\partial y_n'}=\text{constant}. \quad . \quad (4)$$

Now we consider a general mechanical system, of n degrees of freedom, whose potential energy E_p is only a function of the generalized coordinates $q_1 \dots q_2 \dots q_n$, and whose kinetic energy depends on the same coordinates and is a quadratic function of the generalized velocities

$$q_1'=\frac{dq_1}{dt}, \quad q_2'=\frac{dq_2}{dt}\dots q_n'=\frac{dq_n}{dt},$$

so that Euler's identity holds as follows:

$$\sum_{i=1}^{n} q_i'\frac{\partial f}{\partial q_i'}=2E_k. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now we put $f=E_k-E_p$, write t instead of x, q instead of y, and obtain from the equations (1), changing the signs:

$$\frac{\partial E_p}{\partial q_1}-\frac{\partial E_k}{\partial q_1}+\frac{d}{dt}\frac{\partial E_k}{\partial\frac{dq_1}{dt}}=0,$$

$$\frac{\partial E_p}{\partial q_2}-\frac{\partial E_k}{\partial q_2}+\frac{d}{dt}\frac{\partial E_k}{\partial\frac{dq_2}{dt}}=0,$$

$$\vdots$$

$$\frac{\partial E_p'}{\partial q_n}-\frac{\partial E_k}{\partial q_n}+\frac{d}{dt}\frac{\partial E_k}{\partial\frac{dq_n}{dt}}=0.$$

These are the equations of Lagrange. From the equations (2), on the other hand, we obtain

$$\frac{d}{dt}\left[(E_k-E_p)-\frac{dq_1}{dt}\frac{\partial E_k}{\partial\frac{dq_1}{dt}}-\dots\frac{dq_n}{dt}\frac{\partial E_k}{\partial\frac{dq_n}{dt}}\right]-\frac{\partial(E_k-E_p)}{\partial t}=0, \quad . \quad . \quad . \quad (6)$$

or by Euler's identity, and putting $E_k+E_p=E$, the total energy:

$$\frac{dE}{dt}+\frac{\partial(E_k-E_p)}{\partial}=0, \quad . \quad . \quad . \quad . \quad (7)$$

or

$$E+\frac{\partial}{\partial t}\int_{t_0}^{t_1}(E_k-E_p)\,dt=0.$$

If we put $S = \int_{t_0}^{t_1} (E_k - E_p)\,dt$, then we obtain the differential equation of Hamilton-Jacobi :

$$E + \frac{\partial S}{\partial t} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If $\frac{\partial f}{\partial t} = \frac{\partial (E_k - E_p)}{\partial t} = 0$, then we obtain from (4),

$$E_k - E_p - 2E_k = \text{constant},$$

or

$$E_k + E_p = E = \text{constant},$$

which is the principle of conservation of energy.

Summary.

The mechanical equations of Lagrange, of Hamilton-Jacobi, and the principle of conservation of energy have been derived from Euler's differential equation of the calculus of variation.

VII. *On the Combustion of Carbonic Oxide.*—Part I. *By* J. P. Baxter, *B.Sc., James Watt Research Fellow, University of Birmingham* *.

Introduction.

In an earlier paper (Harrison and Baxter, Phil. Mag. Jan. 1927) two methods of investigating gaseous combustion in a closed sphere were described. The first consisted in measuring the relative rise in temperature at a point during the explosion, as indicated by a platinum resistance thermometer, using an Einthoven String Galvanometer as the recording device ; and the second of actually measuring the flame-velocity, using two or more fine wires as the means of recording the passage of the flame.

The first of these methods was applied to a number of

* Communicated by Professor F. W. Burstall, M.Sc., M.A., M.Inst.C.E. M.I.Mech.E.

explosive mixtures, and the following conclusions were arrived at:—

(*a*) The combustion of dry carbon monoxide probably takes place in two ways—firstly by direct oxidation, and secondly by the intervention of water-vapour.

(*b*) The addition of extremely small traces of hydrogen produces a great acceleration in the speed of combustion, and may alter the method of combustion.

With a view to further investigating the above points, it was decided to carry out a series of measurements on explosions of carbon monoxide and air into which small regulated amounts of water-vapour, hydrogen, and other impurities would be introduced.

For this purpose a particularly pure supply of carbon monoxide is required, and owing to the size of the apparatus the gas must be available on a large scale. The department has for some years possessed a plant for the generation of carbon monoxide on a semi-works scale, so in 1926 Mr. Harrison and the author redesigned this apparatus, eliminating all metallic portions and constructing the reaction side entirely in silica and glass. The gas can now be made 99·8 % pure, the impurity being air, though during compression into cylinders the air impurity may go up to 2 %. The important point is that the gas, so far as we can tell, contains no hydrogen.

The Effect of Water-vapour on Carbon-monoxide Explosions.

In 1884, H. B. Dixon published a series of investigations on the action of water-vapour on the carbon-monoxide oxygen reaction. He showed that the mixture $2CO+O_2$, when dried over phosphorus pentoxide, would not combine when sparked in a eudiometer. He also showed that the addition of traces of dry carbon dioxide, cyanogen, air, nitrous oxide, and carbon bisulphate did not cause combination to occur, but traces of water-vapour, ether, pentane, hydrogen chloride, and hydrogen sulphide caused ignition to take place with ease.

Later he measured detonation velocities in the theoretical carbon-monoxide oxygen mixture containing different quantities of water-vapour, and showed that a maximum velocity was reached in the mixture containing 5·6 % water-vapour. Further increase above this value caused a slow decrease in flame-velocity.

Many workers investigated the possibility of igniting dry carbon-monoxide oxygen mixtures, and many theories have been put forward to explain the action of the water-vapour. Recent work by Prof. Bone and his co-workers seems to have established definitely that the direct oxidation of carbon monoxide in flames does take place, but that the presence of water or hydrogen compounds greatly facilitates the combustion. Quantitative measurements on this subject have also been made by R. W. Fenning, using a pressure indicator in a closed vessel, and by Garner and Johnson, who measured the infra-red emission from the flames of wet and dry carbon monoxide.

The present work, of which this is the first part, will endeavour to compare the action of hydrogen, in combination with a number of different elements, on the carbon-monoxide reaction. For this purpose it is intended to measure flame-velocities both directly and by means of a pressure indicator to be described later, and also to compare the time-temperature curves obtained as described in the previous paper. At the start, mixtures with air instead of with oxygen will be used, as these give lower flame-velocities, which are therefore more accurately measured, and also because at the same time it is intended to carry out measurements on the possible "activation of nitrogen" at low pressures.

The Hygroscopic Condition of the Gases.

Although this paper deals only with explosions at one atmosphere initial pressure, the whole of the external apparatus to the sphere is designed to work up to ten atmospheres, as it is intended ultimately to reach this pressure.

The percentage of water-vapour in the gases is regulated by a system of saturation at a given temperature. It is also possible to use "dry" gases which have been passed slowly up a long column of calcium chloride

The saturators consist of steel tubes eight inches long and three and a half inches in diameter, with the ends closed by screwed steel caps, making a gas-tight joint. The gas enters at the bottom through a small diameter (2 mm.) steel pipe, silver soldered through the wall of the saturator, and then enters the water through a number of very fine holes drilled in the sides of the tube. A series of baffles are arranged which ensure the bubbles being well broken up and thoroughly saturated. At the top of the saturator is the

gas outlet with control valve, leading directly to the explosion sphere, and for the pressures used so far a sealed mercury in glass pressure-gauge for the regulation of the pressure inside the saturator.

The whole of this apparatus is immersed in a large water thermostat fitted with two efficient stirrers. An electric immersion heater is controlled by a thermostat regulator of fairly conventional design, the regulator break being connected with a two-volt battery and a quick break relay which operates the main heater. This thermostat has an error not exceeding one-half a degree centigrade.

During the process of filling the sphere the pressure in the saturator is kept constant at the pressure at which the sphere will ultimately be filled, so that the water-vapour content is accurately regulated.

The Pressure Indicator.

As a great deal of constant volume explosion work has been carried out in the past by means of the average pressure determinations, it has been thought desirable to fit a pressure indicator to the sphere for comparison purposes.

One of the latest types of Prof. Burstall's optical indicators has been fitted, modified in such a way as to enable it to be used for sphere work. The diaphragm used is of the flat type, with one corrugation round the circumference. It is of nickel chrome steel, and is machined out of the solid to cut down hysteresis effects. The optical system is of the standard type, except that an oil dashpot device has been arranged to rotate one of the mirrors during the explosion, thus giving a curve recording pressure against time. A time marker, consisting of a small synchronous motor driven by a standard vibrating bar, is arranged to interrupt the source of light every one hundredth of a second, thus giving a suitable time record on the plate. The indicator has been accurately calibrated on the standard mercury column erected for that purpose in the department.

The Recording Circuit.

For velocity measurements the recording circuit is the same as described in the previous paper. Normally two recording wires are used, one one inch from the spark, and the other one inch from the walls. The distance between

the wires is six inches. It is the average flame-velocity over this distance which is measured. Investigations of the way the flame travels over this space have shown that there is a steady diminution in velocity as the flame approaches the walls. This may be due to the increase in initial pressure of the unburnt gas caused by the combustion taking place near the centre, in accordance with the results of Newey and Crowe, who showed that, with increase in initial pressure from one to ten atmospheres, there was a decrease in flame-velocity. On the other hand, the increase in temperature produced by this compression should cause a greater flame-velocity. There is, however, no doubt that a steady diminution in velocity does take place as the flame spreads out over the sphere. In the present case, however, it is the average speed over a length of six inches in a direction inclined at forty-five degrees to the horizontal which is measured and used for purposes of comparison.

For the relative temperature measurements the thermometer as used before has been improved by the addition of a pair of compensating leads which balance out any resistance changes in the supports. It may be stated here that this thermometer gives almost identical results as the one originally used, showing that the effect of the leads, if any, is exceedingly small. The actual recording wire is of platinum-iridium alloy one and a half thousandths of an inch in diameter.

Flame-velocities with varying Mixture-strength and Moisture-content.

A series of determinations were carried out to discover if the alteration in moisture-content produced comparable effects with different mixture-strengths.

The flame-velocities were measured over the available range of mixture-strengths, with water-vapour contents of ·88, 1·2, and 1·6 °/₀, and also with gases slowly dried over calcium chloride. The results are set out in Table I.

These velocities are plotted against mixture-strength in fig. 2. There is no immediately apparent relationship between flame-velocity and moisture-content, the relationship varying with the mixture-strength.

A comparable set of results over a larger range of moisture-content are shown in Table II. and fig. 1. These are arbitrary velocities calculated from the pressure time-

records on the assumption that the flame reaches the wall at the time of maximum pressure. This is not strictly justified, and it is interesting to note that the curves, while similar, are definitely not of the same form, the greatest divergence being in the case of the dry mixtures.

In figs. 3 to 4 are set out a number of relative-temperature time curves, obtained with the compensated thermometer as previously described. Fig. 3 shows the progressive effect of addition of moisture to the $2CO_2 + O_2 + 4N_2$ mixture. The general shape of the curve persists, but the time to reach maximum temperature is steadily reduced. The curve fig. 3 (*a*) is of interest for the somewhat violent temperature oscillations which follow the maximum. Similar changes are apparent on several other curves, and when explosions at higher pressures than these are fired, the oscillations become increasingly noticeable. It was thought that these might be caused by pressure oscillations set up inside the sphere; and to test this a microphone was designed which should record high-frequency pressure-changes over a very small area. It consists of a hollow phosphor-bronze cylinder three inches long by one inch in diameter, carrying at one end a small steel diaphragm three-eighths of an inch in diameter and five thousanths thick. Rigidly attached to the inside of this is a light coil of 40-gauge enamelled wire, arranged so that it can move freely over the head of an electro bar magnet. The leads for the existing coil for the magnet and those from the recording coil are conveyed outside the sphere inside the rods which support the protective casing. An exciting current of half an ampere is used, and the recording coil is connected directly to the Einthoven string galvanometer. When arranged so that the recording diaphragm is in the position usually occupied by the thermometer, a series of pressure oscillations are recorded of identical frequency with the temperature oscillations shown by the thermometer; thus proving the origin of these changes.

Figs. 4 to 9 show the temperature curves for a number of different mixture-strengths containing different amounts of water-vapour.

Fig. 10 shows a reproduction of the pressure curves from the Burstall Optical Indicator for the mixture $2CO + O_2 + 4N_2$ with increasing moisture-content. It is noticeable that, as the time of attainment of maximum pressure decreases, the rate of subsequent cooling tends to increase.

It is not proposed to enter into any further discussion of these results at present, as the work is being continued along the lines of investigating the action of traces of hydrogen and also other compounds of hydrogen, and a general consideration of results is better left until a later paper.

In connexion with the assumption that the combustion of carbon monoxide takes place in two ways, a direct, and an indirect involving the action of water-vapour or some similar body, some interesting analysis has been carried out on the burned gases from the mixtures used in these explosions by the author's colleague Mr. L. E. Winterbottom, who has estimated the oxides of nitrogen produced during the reaction. His results are given at the end of this paper. It will be seen that, as the water-vapour content of the gases increases, the amount of oxides of nitrogen formed decreases very considerably. An increase in moisture-content causes more rapid cooling of the burned gases; and if the nitrogen oxides were the product of a purely thermal change, one would expect them to increase rather than decrease with the addition of water-vapour.

If, however, they are produced as a result of the direct oxidation of carbon monoxide, then the dry gases will favour a maximum yield, and wet gases favouring the indirect oxidation will give a reduced yield of nitrogen oxides. This is in agreement with Bone's figures for oxides of nitrogen produced at high pressures, which are generally recognized as favouring the direct oxidation.

The Estimation of Oxides of Nitrogen.

During the combustion of carbon monoxide and oxygen in the presence of nitrogen, oxidation of the nitrogen may also occur. Bone has shown that as much as 3 per cent. of NO is formed during the combustion of a mixture $2CO + 3O_2 + 2N_2$ at 75 atmospheres original pressure.

In the present research it is proposed to investigate the formation of oxides of nitrogen at low pressures—covering a range from one atmosphere to ten atmospheres original pressure.

The method of estimating small amounts of oxides of nitrogen has been arrived at after considerable experiment, and is essentially a micro-Kjeldhal estimation.

The combustion products, after stirring, are drawn from the explosion sphere by means of a Geryk vacuum pump through a series of absorption tubes containing concentrated potassium-hydroxide solution. Formation of potassium nitrite and potassium nitrate takes place—with nitrite in excess—as established by Le Blanc *.

TABLE I.

Percentage of CO.	Percentage of N_2.	Percentage of O_2.	Percentage of H_2O.	Flame-velocity. cm./sec.
21·0	63·2	15·8	Dry over $CaCl_2$.	59
26·0	59·2	14·8	—	73
30·0	56·0	14·0	—	75
34·0	52·8	13·2	—	77
40·0	48·0	12·0	—	75
48·0	41·6	10·4	—	72
59·0	32·8	8·2	—	52
15·0	68·0	17·0	·88	50
23·0	61·0	15·4	—	95
27·5	58·0	14·5	—	132
29·0	56·8	14·2	—	138
34·0	52·8	13·2	—	165
40·0	48·0	12·0	—	180
45·0	44·0	11·0	—	170
58·0	33·6	8·4	—	127
19·0	64·8	16·2	1·2	85
23·4	61·2	15·3	—	115
29·0	56·8	14·2	—	160
38·0	49·6	12·4	—	197
45·0	44·0	11·0	—	170
63·0	29·6	7·4	—	120
20·0	64·0	16·0	1·6	135
27·5	58·0	14·5	—	190
34·0	52·8	13·2	—	220
40·0	48·0	12·0	—	235
48·0	41·6	10·4	—	220
62·0	29·6	7·4	—	160

* *Zeitsch. Elektrochem.* xii. p. 541 (1906).

After oxidation of the nitrite to nitrate the potassium nitrate is estimated by a micro-modification of Jodlbauer's* method of estimating nitrates. This micro-Kjeldahl method has given consistent results, using direct titration in one hundredth normal solutions. Smaller quantities of ammonia in the distillate may necessitate the use of a colorimetric method for its estimation, but these colour-matching methods have not given accurate results in preliminary experiments.

TABLE II.

Percentage of CO.	Percentage of N_2.	Percentage of O_2.	Percentage of H_2O.	Maximum pressure.	Time to reach max. pressure.	Calculated velocity.
27·0	58·4	14·6	Dry over $CaCl_2$.	88 lb.	·38 sec.	60 cm./sec.
20·0	64·0	16·0		70	1·0	23
35·0	52·0	15·0	—	92	·30	89
48·0	41·6	10·4	—	84	·30	89
40·0	48·0	12·0	—	92	·24	95
58·0	33·6	8·4	—	88	·40	57
28·0	57·6	14·4	·88	100	·19	121
36·0	51·2	12·8	·88	97	·15	154
21·0	63·2	15·8	·88	86	·42	55
43·0	45·6	11·4	·88	100	·14	164
56·0	35·2	8·8	·88	90	·19	121
48·0	41·6	10·4	1·2	95	·12	194
42·0	46·4	11·6	1·2	100	·11	208
31·0	55·2	13·8	1·2	104	·13	177
24·0	60·8	15·2	1·2	92	·19	121
18·0	75·6	16·4	1·2	58	·72	32
36·0	51·2	12·8	1·2	100	·12	194
63·0	29·6	7·4	1·2	72	·19	121
30·0	56·0	14·0	3·1	104	·09	255
40·0	48·0	12·0	3·1	100	·08	290
19·0	64·8	16·2	3·1	85	·19	121
30·0	56·0	14·0	4·4	105	·07	330

* *Z. anal. Chem.* xxvii. p. 92.

TABLE III.

Fig.	No.	Percentage of CO.	Percentage of N_2.	Percentage of O_2.	Percentage of H_2O.
3.	*a*	30	56	14	·88
	b	30	56	14	Dry.
	c	30	56	14	1·2
	d	30	56	14	3·1
	e	30	56	14	4·4
4.	*a*	40	48	12	Dry.
	b	20	64	16	—
5.	*a*	27	58·4	14·6	Dry.
	b	48	41·6	10·4	—
6.	*a*	21	63·2	15·8	·88
	b	43	45·6	11·4	·88
7.	*a*	28	57·6	14·4	·88
	b	56	35·2	8·8	·88
8.	*a*	24	60·8	15·2	1·2
	b	42	46·4	11·6	1·2
9.	*a*	31	55·2	13·8	1·2
	b	48	41·6	10·4	1·2
10.	*a*	30	56	14	4·4
	b	30	56	14	3·1
	c	30	56	14	1·2
	d	30	56	14	·88
	e	30	56	14	Dry.

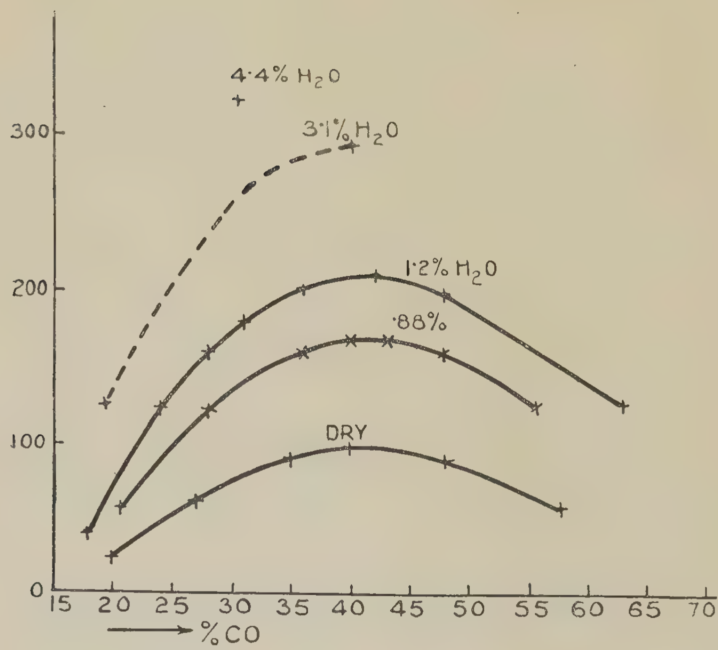
Experiments at one atmosphere original pressure have been conducted, and the effect on NO formation of

(*a*) Percentage of CO in the unfired mixture.
(*b*) Percentage of H_2O in the unfired mixture has been ascertained.

It is evident that formation of NO is favoured in a weak mixture—where excess of oxygen is present.

The presence of water-vapour has a very pronounced effect on oxidation, an addition of 1·2 per cent. of H_2O to a dry 20-per-cent. mixture, for instance, halves the amount of nitrogen oxidation.

Fig. 1.



It is hoped to continue the research with special reference to—

(1) Effect of small quantities of hydrogen on NO formation.

(2) Formation of NO in CO-O mixtures containing small quantities of nitrogen.

(3) Effect of addition of oxygen to a constant mixture of nitrogen and carbon monoxide.

Fig. 2.

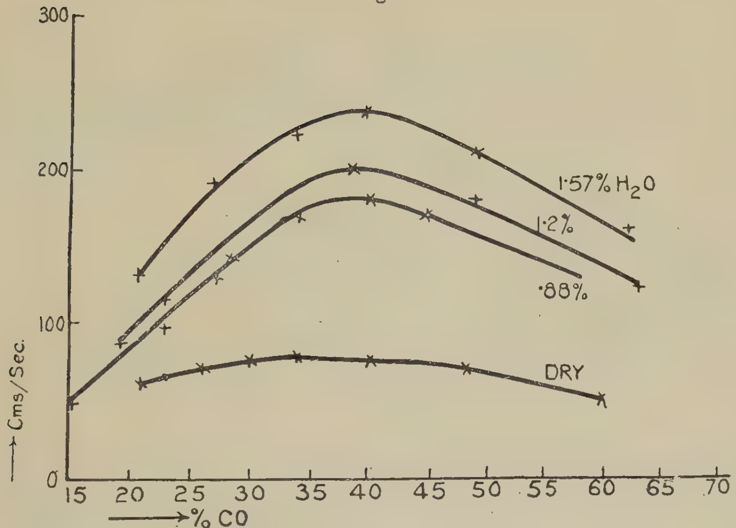


Fig. 3.

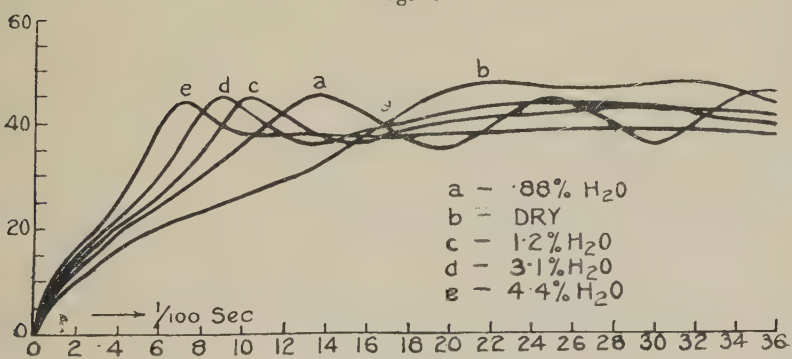


Fig. 4.

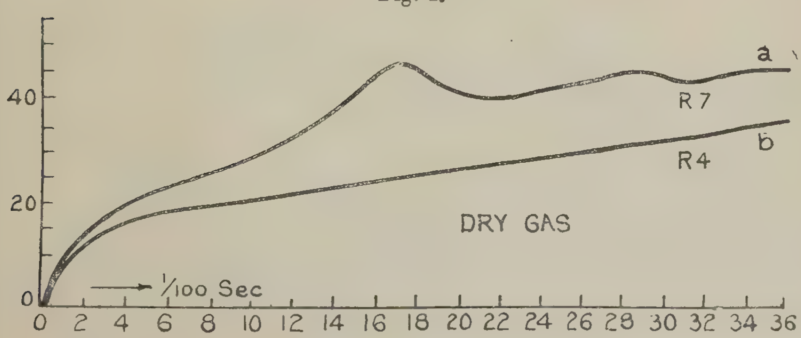


Fig. 5.

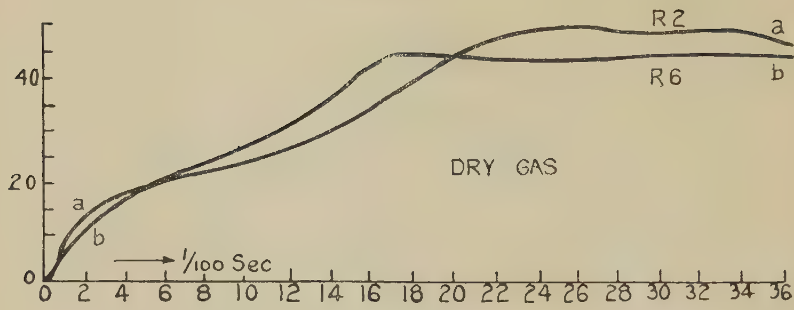


Fig. 6.

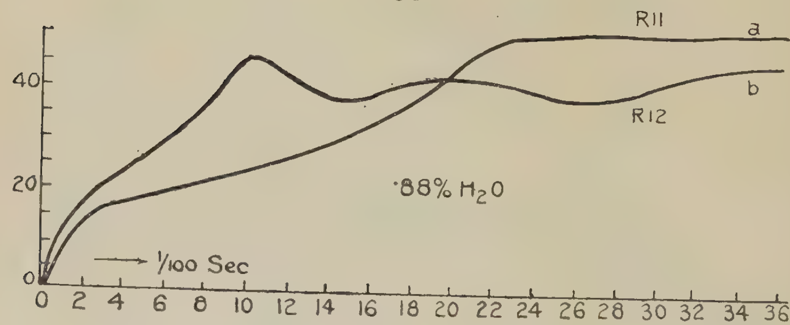


Fig. 7.

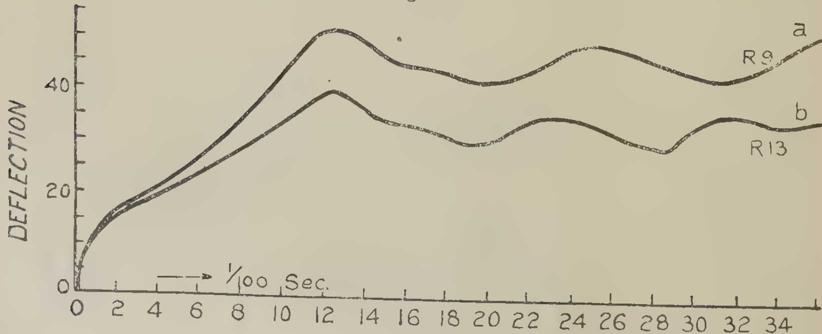


Fig. 8.

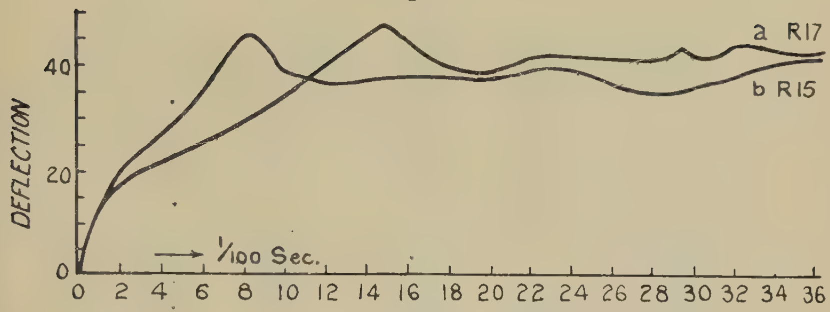


Fig. 9.

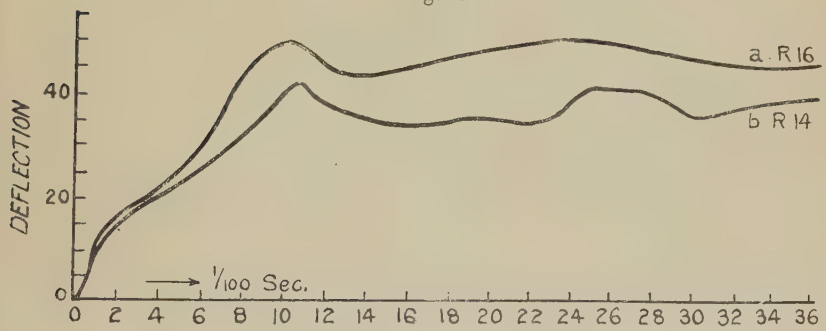


Fig. 10.

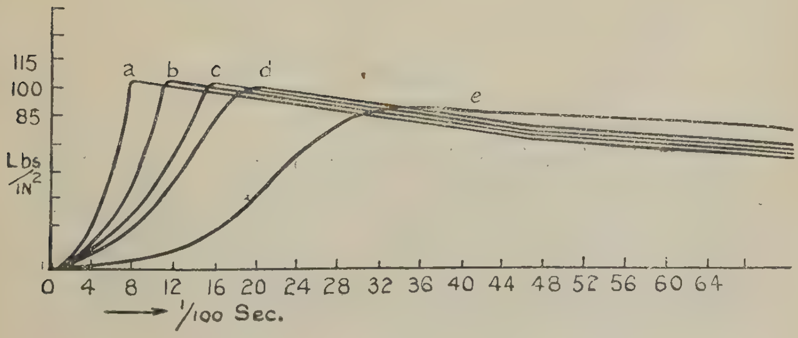
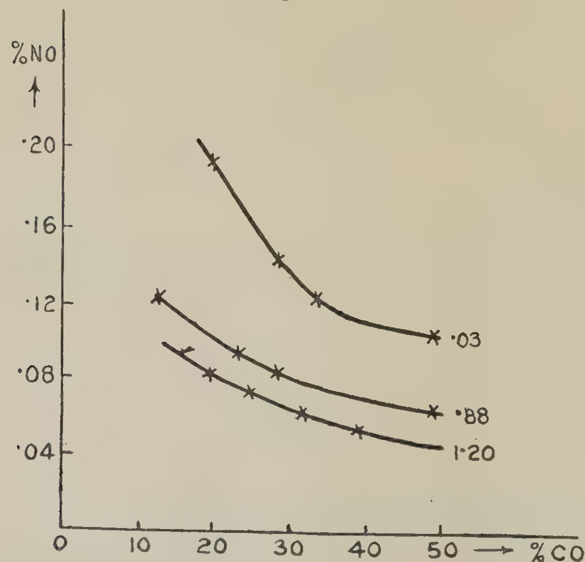


TABLE IV.

Percentage of CO.	Percentage of N.	Percentage of O.	Percentage of H_2O.	Percentage of NO.
27	58·4	14·6	·03	·14
19	64·8	16·2	·03	·19
35·5	51·6	12·9	·03	·12
48	41·6	10·4	·03	·10
13	70·4	17·6	·88	·12
45	44	11	·88	·06
27·6	58	14·5	·88	·08
23	61·6	15·4	·88	·09
38·3	49·6	12·3	1·2	·05
24·6	60·4	15·1	1·2	·07
19·0	64·6	16·2	1·2	·08
31	55·2	13·6	1·2	·06

Fig. 11.



In conclusion, the author wishes to express his gratitude to Professor Burstall for much helpful advice, and to Mr. Pickering, of Birmingham University, for invaluable assistance in the construction and assembling of the apparatus required. He also thanks the Department of Scientific and Industrial Research for a grant which has defrayed the expenses of the research.

VIII. *The Torsion-Flexure Oscillations of a System of Two Connected Beams.* By S. B. Gates, *M.A.**

1. THE ordinary theory of the independent flexural and torsional oscillations of bars applies in strictness only to movement of the system *in vacuo*, since only the elastic and inertia forces are included. If the oscillation takes place in a fluid medium, the fluid reactions provide another system of forces which may or may not be negligible compared with those usually considered. If the system as a whole has a large rate of translation through the fluid, it is probable that the fluid reactions during the oscillation become important. For instance, it may be necessary to retain the aerodynamic forces when considering the vibration of an elastic component of an aeroplane in flight. A type of elastic hydrodynamic problem is thus presented which assumes some importance in aeronautics, and has not hitherto received much study. One important consequence of the retention of the fluid reaction is that the independence of the flexural and torsional oscillations is destroyed. Thus if z is the displacement of the central line and θ the twist of the beam, the equations to be solved are of the form:

$$\frac{d^2z}{dt^2}+a^2\frac{d^4z}{dx^4}+f(z,\theta)=0,$$

$$\frac{d^2\theta}{dt^2}-b^2\frac{d^2\theta}{dx^2}+\phi(z,\theta)=0,$$

where $f(z,\theta)$ and $\phi(z,\theta)$, representing the fluid reactions, will usually comprise terms proportional to $\frac{dz}{dt}$, $\frac{d\theta}{dt}$, and θ.

The exact solution of these equations is of the form:

$$z=\Sigma\Sigma\,\alpha e^{\lambda x}e^{\mu t},$$

$$\theta=\Sigma\Sigma\beta e^{\lambda x}e^{\mu t},$$

and if we proceed in the usual way to find the admissible values of λ and μ, we are led to a cubic equation in λ^2, the coefficients being, of course, functions of μ. It appears to be very difficult to proceed algebraically with the exact solution beyond this stage.

2. It seems, therefore, that the problem of the related

* Communicated by R. V. Southwell, M.A., F.R.S.

flexural and torsional oscillations of a long beam under the action of "external" forces imposed by the surrounding fluid is a complex and difficult one. There is, however, an elastic system of simpler type which, under certain simplifying assumptions, yields an exact algebraic solution, and this has features of practical and theoretical interest which may be put on record. We shall consider two long beams or spars of equal length and constant, but not necessarily the same, cross-section, each encastred at one end and free both to deflect in a direction perpendicular to the line joining the fixed ends and to twist about its central line. The spars are continuously connected along their length by connexions flexible enough to allow them to execute the prescribed motion. This system is supposed to oscillate in a fluid medium which gives rise to reactions of the type specified in the preceding paragraph. The following assumptions are made:

(*a*) All stresses in the connexions are neglected.
(*b*) The torsional rigidity of each spar is neglected, so that the torsional rigidity of the whole system arises solely from the flexural rigidity of the spars.

The centre of mass of a cross-section of this system, which in its undisplaced position is perpendicular to the spars, has a velocity parallel to the direction of motion of each spar, and the cross-section is twisting under the difference of the deflexions of the two spars. Hence the system may be regarded as a single "beam" of elongated cross-section which is executing combined flexural and torsional oscillations. Under the restrictions (*a*) and (*b*) the tractions across this section are supposed to be concentrated at the two spar centres and to consist of shear forces due to the separate deflexion of each spar. It is not necessary to assume that every dimension of this cross-section is small compared with the length of the spars.

On the practical side the system here contemplated represents an attempt to abstract the essential elastic features of a more or less flat rectangular structure, like an aeroplane wing, which is built up on two spars each encastred at one end. It is obvious that the strength of such a structure will reside principally in its spars, and also that large torsional rigidity of the spars themselves is not in general necessary to ensure torsional stiffness of the whole structure. Thus the assumptions (*a*) and (*b*) above, which have been introduced to bring the problem within the scope of an exact solution, are not such as to vitiate entirely its application to

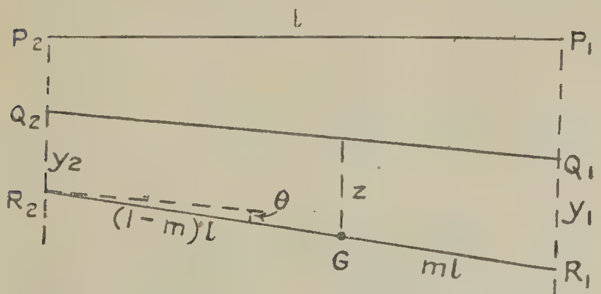
such subjects as the vibration of a cantilever monoplane wing in a stream of air.

3. Proceeding with the problem under the above assumptions, the following is the notation :—

x coordinate along a spar.
$x=0$ at encastred end.
$x=r$ at free end.
l distance between spar centres.
m the centre of mass of a cross-section of the structure divides l in the ratio $m : 1-m$.
y_1, y_2 deflexions of the spar centres at section x, measured from their equilibrium positions.
z deflexion of centre of mass of section at x, measured from its equilibrium position.
θ twist of section x, measured from its equilibrium position.
M mass of structure per unit of x.
I moment of inertia of structure, per unit of x, about an axis through the centre of mass of a section and parallel to the spars.
B_1, B_2 flexural rigidities of spars.
$f(z, \theta)$ component of fluid reaction parallel to z, per unit of x.
$\phi(z, \theta)$ moment of fluid reaction about centre of mass of section per unit of x.

Fig. 1.

Diagram illustrating the motion of a section of the structure.



P_1, P_2 are the spar centres at the encastred end $x=0$.
Q_1, Q_2 are the spar centres at section x in equilibrium.
R_1, R_2 are the spar centres at section x during the oscillation.
G is the centre of mass of section x.

A section of thickness dx is in motion under shear forces at R_1 and R_2 and the fluid reactions. Resolving parallel to z and taking moments about G, we have, remembering that all coordinates are measured from the equilibrium position of the section:

$$\mathrm{M}dx\,.\frac{d^2z}{dt^2}=-\mathrm{B}_1\frac{d^4y_1}{dx^4}dx-\mathrm{B}_2\frac{d^4y_2}{dx^4}dx+f\,dx,$$

$$\mathrm{I}dx\,.\frac{d^2\theta}{dt^2}=-ml\left(\mathrm{B}_1\frac{d^4y_1}{dx^4}dx\right)+(1-m)l\left(\mathrm{B}_2\frac{d^4y_2}{dx^4}dx\right)+\phi\,dx.$$

Since
$$y_1=z+ml\theta,$$
$$y_2=z-(1-m)l\theta,$$

we eliminate y_1, y_2 to obtain:

$$\mathrm{M}\frac{d^2z}{dt^2}=-\mathrm{B}_1\left(\frac{d^4z}{dx^4}+ml\frac{d^4\theta}{dx^4}\right)-\mathrm{B}_2\left(\frac{d^4z}{dx^4}-\overline{1-m}\,l\frac{d^4\theta}{dx^4}\right)+f,$$

$$\mathrm{I}\frac{d^2\theta}{dt^2}=-ml\mathrm{B}_1\left(\frac{d^4z}{dx^4}+ml\frac{d_4\theta}{dx^4}\right)$$
$$+(1-m)l\mathrm{B}_2\left(\frac{d^4z}{dx^4}-\overline{1-m}\,l\frac{d^4\theta}{dx^4}\right)+\phi.$$

If, as is generally the case, the fluid reactions due to the oscillation are the sums of terms in $\frac{dz}{dt}$, $\frac{d\theta}{dt}$, and θ, we may write:

$$f=-\mathrm{M}\left(a_1\frac{dz}{dt}+b_1\frac{d\theta}{dt}+c_1\theta\right),$$

$$\phi=-\mathrm{I}\left(a_2\frac{dz}{dt}+b_2\frac{d\theta}{dt}+c_2\theta\right).$$

The equations of motion now reduce to:

$$\left.\begin{aligned}\frac{d^2z}{dt^2}+\alpha_1\frac{d^4z}{dx^4}+\beta_1\frac{d^4\theta}{dx^4}+a_1\frac{dz}{dt}+b_1\frac{d\theta}{dt}+c_1\theta=0\\ \frac{d^2\theta}{dt^2}+\alpha_2\frac{d^4\theta}{dx^4}+\beta_2\frac{d^4z}{dx^4}+a_2\frac{dz}{dt}+b_2\frac{d\theta}{dt}+c_2\theta=0\end{aligned}\right\}, \quad . \quad (1)$$

where
$$\alpha_1=\frac{\mathrm{B}_1+\mathrm{B}_2}{\mathrm{M}},$$

$$\beta_1=\frac{l[m\mathrm{B}_1-(1-m)\mathrm{B}_2]}{\mathrm{M}},$$

$$\alpha_2=\frac{l^2[m^2\mathrm{B}_1+(1-m)^2\mathrm{B}_2]}{\mathrm{I}},$$

$$\beta_2=\frac{l[m\mathrm{B}_1-(1-m)\mathrm{B}_2]}{\mathrm{I}}.$$

4. To solve equations (1) we assume in the usual way:

$$z = g(x)e^{\mu t},$$
$$\theta = h(x)e^{\mu t}.$$

Then, writing D for $\frac{d}{dx}$, the equations for g and h are:

$$(\alpha_1 D^4 + \mu^2 + a_1\mu)g + (\beta_1 D^4 + b_1\mu + c_1)h = 0,$$
$$(\beta_2 D^4 + \mu a_2)g + (\alpha_2 D^4 + \mu^2 + b_2\mu + c_2)h = 0.$$

Hence, if $g = Ae^{\lambda x}$, $h = Be^{\lambda x}$, we have:

$$A(\alpha_1\lambda^4 + \mu^2 + a_1\mu) + B(\beta_1\lambda^4 + b_1\mu + c_1) = 0,$$
$$A(\beta_2\lambda^4 + a_2\mu) + B(\alpha_2\lambda^4 + \mu^2 + b_2\mu + c_2) = 0,$$

whence

$$k = \frac{B}{A} = -\frac{\alpha_1\lambda^4 + \mu^2 + a_1\mu}{\beta_1\lambda^4 + b_1\mu + c_1} = -\frac{\beta_2\lambda^4 + a_2\mu}{\alpha_2\lambda^4 + \mu^2 + b_2\mu + c_2}. \quad (2)$$

Thus a solution of equation (1) is:

$$\left.\begin{aligned} z &= \Sigma\Sigma\, Ae^{\lambda x}e^{\mu t} \\ \theta &= \Sigma\Sigma k A e^{\lambda x}e^{\mu t} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where k is given by equation (2), and one relation between λ and μ, given by the last equality in (2), is found to be:

$$(\alpha_1\alpha_2 - \beta_1\beta_2)\lambda^8 + \{(\alpha_1 + \alpha_2)\mu^2 + (a_1b_2 - \beta_1\alpha_2 + \alpha_2 a_1 - \beta_2 b_1)\mu$$
$$+ (\alpha_1 c_2 - \beta_2 c_1)\}\lambda^4 + \mu^4 + (a_1 + b_2)\mu^3$$
$$+ (a_1b_2 - a_2b_1 + c_2)\mu^2 + (a_1c_2 - a_2c_1)\mu = 0. \quad . \quad (4)$$

This, being a quadratic in λ^4, allows of further progress. It may be noticed that retention of the torsional stiffness of the spars would introduce a term $\frac{d^2\theta}{dx^2}$ in the second equation of (1), and thence terms in λ^6 and λ^2 in (4).
This would destroy the possibility of an algebraic solution.

5. The second equation which, with (4), will determine all the admissible values of λ and μ in (3), is to be found from the end conditions of the spars.

Since the spars are encastred at $x=0$ and free at $x=r$, we have:

$$y_1 = \frac{dy_1}{dx} = y_2 = \frac{dy_2}{dx} = 0 \text{ at } x=0,$$

and $$\frac{d^2y_1}{dx^2}=\frac{d^3y_1}{dx^3}=\frac{d^2y_2}{dx^2}=\frac{d^3y_2}{dx^3}=0 \text{ at } x=r,$$

for all values of t.

These lead to

$$\left.\begin{array}{ll} & z=\dfrac{dz}{dx}=\theta=\dfrac{d\theta}{dx}=0 \text{ at } x=0, \\ \text{and} & \dfrac{d^2z}{dx^2}=\dfrac{d^3z}{dx^3}=\dfrac{d^2\theta}{dx^2}=\dfrac{d^3\theta}{dx^3}=0 \text{ at } x=r, \end{array}\right\}, \qquad (5)$$

for all values of t.

If for a value of μ which is admissible in (3) the roots of (4) are $\lambda_1, \lambda_2, \ldots \lambda_8$, and if the constants A are $A_1, A_2, \ldots A_8$, the conditions (5) require that:

$$\left.\begin{array}{ll} \sum\limits_{n=1}^{n=8} A_n & =0 \\ \Sigma\, k_n A_n & =0 \\ \Sigma\, \lambda_n A_n & =0 \\ \Sigma\, k_n\lambda_n A_n & =0 \\ \Sigma\, \lambda_n^2 e^{r\lambda_n} A_n & =0 \\ \Sigma\, k_n\lambda_n^2 e^{r\lambda_n} A_n & =0 \\ \Sigma\, \lambda_n^3 e^{r\lambda_n} A_n & =0 \\ \Sigma\, k_n\lambda_n^3 e^{r\lambda_n} A_n & =0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Elimination of the ratios $A_1 : A_2 : \ldots A_8$ from the eight equations (6) leads to an eighth row determinantal equation involving λ and μ. This, together with (4), suffices to determine all the admissible values of λ and μ.

Finally, the single constant A which is undetermined by (6) for given values of λ and μ must be found from the conditions at $t=0$. This completes the formal solution of the problem.

6. *Reduction of the Determinantal Equation.*

The roots of equation (4) are given by $\lambda^4=\lambda_1^4$ and $\lambda^4=\lambda_2^4$, where λ_1^4 and λ_2^4 aro complex in general. Hence the roots are

$$\pm\lambda_1, \quad \pm i\lambda_1, \quad \pm\lambda_2, \quad \pm i\lambda_2,$$

and it is clear from the expression (2) for k that those roots with suffix 1 will have a single value of k, say k_1, associated with them, and that a single k_2 will be associated with those of suffix 2.

The determinant arising from (6) is therefore :

$$\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
k_1 & k_1 & k_1 & k_1 & k_2 & k_2 & k_2 & k_2 \\
\lambda_1 & -\lambda_1 & i\lambda_1 & -i\lambda_1 & \lambda_2 & -\lambda_2 & i\lambda_2 & -i\lambda_2 \\
k_1\lambda_1 & -k_1\lambda_1 & ik_1\lambda_1 & -ik_1\lambda_1 & k_2\lambda_2 & -k_2\lambda_2 & ik_2\lambda_2 & -ik_2\lambda_2 \\
\lambda_1^2e^{r\lambda_1} & \lambda_1^2e^{-r\lambda_1} & -\lambda_1^2e^{ir\lambda_1} & -\lambda_1^2e^{-ir\lambda_1} & \lambda_2^2e^{r\lambda_2} & \lambda_2^2e^{-r\lambda_2} & -\lambda_2^2e^{ir\lambda_2} & -\lambda_2^2e^{-ir\lambda_2} \\
k_1\lambda_1^2e^{r\lambda_1} & k_1\lambda_1^2e^{-r\lambda_1} & -k_1\lambda_1^2e^{ir\lambda_1} & -k_1\lambda_1^2e^{-ir\lambda_1} & k_2\lambda_2^2e^{r\lambda_2} & k_2\lambda_2^2e^{-r\lambda_2} & -k_2\lambda_2^2e^{ir\lambda_2} & -k_2\lambda_2^2e^{-ir\lambda_2} \\
\lambda_1^3e^{r\lambda_1} & -\lambda_1^3e^{-r\lambda_1} & -i\lambda_1^3e^{ir\lambda_1} & i\lambda_1^3e^{-ir\lambda_1} & \lambda_2^3e^{r\lambda_2} & -\lambda_2^3e^{-r\lambda_2} & -i\lambda_2^3e^{ir\lambda_2} & i\lambda_2^3e^{ir\lambda_2} \\
k_1\lambda_1^3e^{r\lambda_1} & -k_1\lambda_1^2e^{-r\lambda_1} & -ik_1\lambda_1^3e^{ir\lambda_1} & ik_1\lambda_1^3e^{-ir\lambda_1} & k_2\lambda_2^3e^{r\lambda_2} & -k_2\lambda_2^3e^{-r\lambda_2} & -ik_2\lambda_2^3e^{ir\lambda_2} & ik_2\lambda_2^{3ir\lambda_2}
\end{vmatrix}$$

This may be reduced without much difficulty to the form:

$$(k_2-k_1)^4 \,.\, \mathrm{F}(r\lambda_1) \,.\, \mathrm{F}(r\lambda_2),$$

where

$$\mathrm{F}(r\lambda)=\lambda^6 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ e^{r\lambda} & e^{-r\lambda} & -e^{ir\lambda} & -e^{-ir\lambda} \\ e^{r\lambda} & -e^{-r\lambda} & -ie^{ir\lambda} & ie^{-ir\lambda} \end{vmatrix}$$

$$=16i\lambda^6(\cosh r\lambda \cos r\lambda + 1).$$

Thus, omitting constant factors, the second equation for the determination of λ and μ is

$$(k_1-k_2)^4\lambda_1^6\lambda_2^6(\cosh r\lambda_1 \cos r\lambda_1 + 1)(\cosh r\lambda_2 \cos r\lambda_2 + 1)=0, \quad . \;\; . \;\; . \quad (7)$$

where λ_1^4, λ_2^4 are the roots of (4).

It is notable that the equation

$$\cosh r\lambda \cos r\lambda + 1 = 0$$

is that which determines the modes in the flexural oscillations of a single long beam encastred at one end and fixed at the other.

7. *Discussion of the Modes.*

It thus appears that if λ_1^4, λ_2^4 are the roots of (4), admissible solutions arise in the following cases:—

(1) $\cosh r\lambda_1 \cos r\lambda_1 + 1 = 0$,
(2) $\cosh r\lambda_2 \cos r\lambda_2 + 1 = 0$,
(3) $k_1 = k_2$,
(4) $\lambda_1 = 0$,
(5) $\lambda_2 = 0$,

or when two or more of these conditions are simultaneously satisfied. Of these, the first two furnish the main part of the solution, the others requiring, in general, the satisfaction of special relations between the constants of the system.

Cases (1) *and* (2).

If z is a root of $\cosh z \cos z + 1 = 0$, then $-z$, iz, and $-iz$ are roots, but there are no complex roots *.

* This may be proved analytically by pedestrian methods, or appeal can be made to the fact that the equation governs the flexural oscillations of a uniform cantilever bar, a motion which is necessarily pure harmonic, since it is not subject to dissipation of energy. (Rayleigh, 'Sound,' i. §§ 173, 175.)

The positive real roots are the infinite series:

$$1{\cdot}875,\ 4{\cdot}69,\ 7{\cdot}85,\ 11{\cdot}00,\ 14{\cdot}14,\ 17{\cdot}28\ \ldots\ldots,$$

approximating to $(n+\frac{1}{2})\pi$ when n is large.

The method of constructing the modes which arise from this equation is therefore as follows:—If $r\lambda_1$ takes any one of these values, the corresponding values of μ, four in general, are obtained by substitution of $\lambda_1{}^4$ in equation (4). With any one of these values of μ, $\lambda_2{}^4$ is now obtained from (4) and the corresponding mode will be

$$\left.\begin{aligned} z&=(\mathrm{A}_1\cosh\lambda_1x+\mathrm{A}_2\sinh\lambda_1x+\mathrm{A}_3\cos\lambda_1x+\mathrm{A}_4\sin\lambda_1x\\ &\quad+\mathrm{A}_5\cosh\lambda_2x+\mathrm{A}_6\sinh\lambda_2x+\mathrm{A}_7\cos\lambda_2x+\mathrm{A}_8\sin\lambda_2x)e^{\mu t}\\ \theta&=\{k_1(\mathrm{A}_1\cosh\lambda_1x+\mathrm{A}_2\sinh\lambda_1x+\mathrm{A}_3\cos\lambda_1x+\mathrm{A}_4\sin\lambda_1x)\\ &\quad+k_2(\mathrm{A}_5\cosh\lambda_2x+\mathrm{A}_6\sinh\lambda_2x+\mathrm{A}_7\cos\lambda_2x+\mathrm{A}_8\sin\lambda_2x)\}e^{\mu t}\end{aligned}\right\}, \quad (8)$$

where μ, λ_2, and the constants may be complex.

In general λ_2 will not satisfy $\cosh r\lambda\cos r\lambda+1=0$, but will be a function of the constants of the system.

It will be noticed that from a single solution of $\cosh z\cos z+1=0$ there arise four different solutions of the type (8), since with each of the four values of μ associated with λ_1 there will be a different λ_2. If there is a pair of complex values of μ, the two corresponding solutions will, of course, combine to give a single real mode.

Hence, of any mode of this series, part takes the same form as in the problem of the single beam encastred at one end, and depends only on the length of the beam and the end conditions, while the other part depends on the arrangement of the two-beam system and the external forces.

Case (3). $k_1=k_2$

In any mode which satisfies this condition we shall have

$$z=f(x)e^{\mu t},$$
$$\theta=\mathrm{C}f(x)e^{\mu t},$$

where C is a constant, and it is evident that the corresponding mode of the deflexions y_1, y_2 of the spars will be of this form. Thus any mode of this class is characterized by the relation

$$y_1/y_2=\text{constant}.$$

To investigate the possibility of such modes we must

return to equation (2), in which, putting $k_1 = k_2 = k$, we have:

$$k(\beta_1\lambda_1^4 + b_1\mu + c_1) + \alpha_1\lambda_1^4 + \mu^2 + a_1\mu = 0,$$

$$k(\beta_1\lambda_2^4 + b_1\mu + c_1) + \alpha_1\lambda_2^4 + \mu^2 + a_1\mu = 0,$$

$$k(\alpha_2\lambda_1^4 + \mu^2 + b_2\mu + c_2) + \beta_2\lambda_1^4 + a_2\mu = 0,$$

$$k(\alpha_2\lambda_2^4 + \mu^2 + b_2\mu + c_2) + \beta_2\lambda_2^4 + a_2\mu = 0.$$

Subtracting the second from the first and the fourth from the third of these equations, we have

$$(k\beta_1 + \alpha_1)(\lambda_1^4 - \lambda_2^4) = 0$$

and
$$(k\alpha_2 + \beta_2)(\lambda_1^4 - \lambda_2^4) = 0,$$

whence it follows that either

$$\lambda_1^4 = \lambda_2^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

or
$$\left.\begin{aligned} k\beta_1 + \alpha_1 &= 0 \\ k\alpha_2 + \beta_2 &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Considering first the alternative (b), it is evident from the four equations above that we must also have:

$$k(b_1\mu + c_1) + \mu^2 + a_1\mu = 0,$$

$$k(\mu^2 + b_2\mu + c_2) + a_2\mu = 0.$$

Thus in case (b) we have

$$-k = \frac{\alpha_1}{\beta_1} = \frac{\beta_2}{\alpha_2} = \frac{\mu^2 + a_1\mu}{b_1\mu + c_1} = \frac{a_2\mu}{\mu^2 + b_2\mu + c_2}.$$

This involves relations between the constants of the system, and it is easily seen that these lead to a violation of the fundamental condition of small deformation, irrespective of the time factor. For under these relations the coefficient of λ^8 and the absolute term in equation (4) both vanish. This implies that one of λ_1, λ_2 tends to zero, the other to infinity. Thus the alternative (b) does not contribute a new type of solution.

The condition of equal roots.—Returning to the alternative (a), $\lambda_1 = \lambda_2$, it is sufficient to remark that our derivation of the determinantal equation becomes invalid if any two values of λ associated with a given μ are equal. For if $\lambda_1 = \lambda_2$, the expression $\mathrm{A}e^{\lambda_1 x} + \mathrm{B}e^{\lambda_2 x}$ must be replaced by $e^{\lambda_1 x}(\mathrm{A}' + \mathrm{B}'x)$ and the expressions (6) for the end conditions become correspondingly modified. Thus the condition $\lambda_1 = \lambda_2$ must

be rejected, without prejudice, however, to the question as to whether equal roots do in fact arise.

It is evident that equal values of λ can arise only through the satisfaction of certain relations between the constants of the system. For such roots and their associated time factors must satisfy not only equation (5), but also the condition that (5) has equal roots. We thus have sufficient equations to determine μ and the repeated λ roots before introducing the end conditions, and the satisfaction of the latter must involve special relations between the constants of the system. It is possible, of course, starting with λ^4 and $(\lambda+d\lambda)^4$, and proceeding to the limit when $d\lambda \rightarrow 0$, to obtain the modified end conditions applicable to repeated roots, and so derive the determinantal equation for this class of solution. This equation is, however, one of great complexity, and the matter, which is of academic interest only, has not been carried further.

Cases (4) *and* (5).

If $\lambda_1=0$ and $k_1 \neq k_2$, then reference to equations (6) shows that λ_2 must be a root of $\cosh r\lambda \cos r\lambda+1=0$. Thus this case is included under Case (2).

Summary.—The survey of Cases (3) to (5) has not yielded anything of practical importance. Thus, in general, the system moves in modes of the type:

$$(A_1 \cosh \lambda_1 x + A_2 \sinh \lambda_1 x + A_3 \cos \lambda_1 x + A_4 \sin \lambda_1 x$$
$$+ A_5 \cosh \lambda_2 x + A_6 \sinh \lambda_2 x + A_7 \cos \lambda_2 x + A_8 \sin \lambda_2 x) e^{\mu t},$$

where λ_1 satisfies $\cosh r\lambda \cos r\lambda + 1 = 0$

and λ_2 and μ are given by equation (4).

8. *The Stability of the System.*

When the motion of an elastic structure takes place in contact with a source of energy such as a fluid medium in motion, the practical problem is to determine under what conditions it becomes unstable. There is a continued interchange of energy between the vibrating structure and the fluid. If on the whole the structure absorbs energy from the fluid its kinetic energy will increase until the elastic limit of the material is passed, and structural failure will follow. Under these conditions the vibration is unstable.

On the other hand, if energy is on the whole dissipated from the structure to the fluid, the structure ultimately comes to rest and the motion is stable. In the intermediate case, when no energy flows on the whole between the structure and the fluid, a condition of permanent oscillation is theoretically attained, and the stability is neutral.

In examining the stability we are only concerned with the values of the time factor μ. The condition for stability is that the real part of every value of μ shall be negative. The stability analysis is relatively simple because all the values of μ are derived from those of λ_1 satisfying

$$\cosh r\lambda \cos r\lambda + 1 = 0,$$

and are independent of the complementary series λ_2, the derivation of which makes the construction of the modes a very laborious process. We have to examine the quartic :

$$\mu^4 + C_1\mu^3 + (C_2\lambda^4 + C_3)\mu^2 + C_4\lambda^4\mu + C_5\lambda^8 + C_6\lambda^4 = 0,$$

where, as will be seen from (4),

$$\begin{aligned} C_1 &= a_1 + b_2, \\ C_2 &= \alpha_1 + a_2. \\ C_3 &= a_1b_2 - a_2b_1 + c_2, \\ C_4 &= \alpha_1b_2 - \beta_1a_2 + \alpha_2a_1 - \beta_2b_1, \\ C_5 &= \alpha_1\alpha_2 - \beta_1\beta_2, \\ C_6 &= \alpha_1c_2 - \beta_2c_1, \end{aligned}$$

and $r\lambda$ has the infinite series of values, already mentioned, which satisfy

$$\cosh z \cos z + 1 = 0.$$

In this we have put $a_1c_2 - a_2c_1 = 0$, a condition which usually holds in the case of fluid reactions.

9. *The Conditions of Stability.*

The conditions of stability of a quartic are that each coefficient, and also Routh's discriminant, must be positive. We must therefore have :

$$\begin{aligned} C_1 &> 0, \\ C_2\lambda^4 + C_3 &> 0, \\ C_4 &> 0, \\ C_5\lambda^4 + C_6 &> 0, \end{aligned}$$

and $$C_1C_4\lambda^4(C_2\lambda^4 + C_3) - C_4^2\lambda^8 - C_1^2(C_5\lambda^8 + C_6\lambda^4) > 0.$$

The last of these reduces to :

$$\Delta = X\lambda^4 + Y > 0,$$

where
$$X = C_1C_2C_4 - C_4^2 - C_1^2C_5,$$
$$Y = C_1C_3C_4 - C_1^2C_6.$$

These conditions must be satisfied for each of the infinite series of values of λ given by

$$r\lambda = 1{\cdot}875,\ 4{\cdot}69,\ 7{\cdot}85, \ldots\ldots (n+\tfrac{1}{2})\pi,$$

when n is large.

The complete analysis of these conditions would require a detailed consideration of the coefficients a_1, b_1, c_1, a_2, b_2, c_2, which specify the fluid reaction. These depend in general in magnitude and sign on the angle of presentation of the structure to the fluid in the equilibrium position ; analysis on these lines is beyond the scope of this paper. It should be remarked, however, that each of these coefficients is proportional to V, the equilibrium velocity of the structure through the fluid. A few results of general import in the examination of stability may be mentioned.

(a) *Two types of instability.*—It appears that a distinction of some importance may be drawn between the conditions of stability. Consider a condition of the form $H\lambda^4 + J > 0$. If $H > 0$ the condition is satisfied when λ is large enough, however large and negative J may be. In this case the condition may be written $H\lambda_0^4 + J > 0$, where λ_0 is the least value of λ ; and if this is not satisfied a limited number of the gravest modes will be unstable. We may call such a condition one of *limited instability.*

On the other hand, if $H < 0$, all the modes involving λ greater than a certain value will be unstable, however large and positive J may be. This may be called a state of *unlimited instability.*

(b) *A short scheme of stability criteria.*—Of the coefficients C, it will be seen that C_2 and C_5 are always positive, while the sign of the others, and of X and Y, depends on the arrangement of the system.

Thus if, as usually happens, C_1 is positive, we may group the stability conditions thus :

$$\left.\begin{array}{r} C_2\lambda_0^4 + C_3 > 0 \\ C_5\lambda_0^4 + C_6 > 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\left.\begin{array}{r} C_4 > 0 \\ X > 0 \\ X\lambda_0^4 + Y > 0 \end{array}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where λ_0 is the smallest value of λ.

Violation of criteria (9) leads to limited instability. Violation of the first two criteria (10) produces unlimited instability, but if $X > 0$, violation of the third of (10) produces limited instability. It is noticeable that if $C_4 < 0$ then $X < 0$, but if $C_4 > 0$ it does not follow that $X > 0$.

The conditions (9) and (10), together with $C_1 > 0$, comprise the shortest complete criterion of stability.

(c) *The influence of beam-length, total strength, and speed.*—Assuming that C_4 and X are both positive, so that only limited instability is in question, the influence of r, $S = B_1 + B_2$, and the steady speed V can be at once seen. Under these conditions, stability can be assured :

(1) by shortening the length of beam r, for this increases λ_0 ;

(2) by increasing the total strength S, for C_2, C_5, and X each involve positive powers of S ;

(3) by decreasing the speed V, which occurs as a factor only of C_3, C_6, and Y—this statement is only true if one of C_3, C_6, and Y is negative, and amounts to saying that if instability of the limited type is latent in the system it will be developed by increasing the steady speed of the structure through the fluid.

(d) *The first appearance of instability.*—It is often necessary to trace the changes which, when applied to a system which is known to be stable, will lead to instability. In this connexion it is only necessary to notice that :

(1) instability arises through the change from negative to positive of the real part of a complex value of μ when Routh's discriminant vanishes ;

(2) instability arises through the change from negative to positive of a real value of μ when the coefficient of the absolute term of the quartic vanishes.

Hence the equations which determine the first appearance of instability in a stable system of the form considered here are :

$$X\lambda_0^4 + Y = 0$$

and

$$C_5\lambda_0^4 + C_6 = 0.$$

10. *The Character of the Higher Modes.*

When λ is large the equation for μ becomes :

$$\mu^4 + C_1\mu^3 + C_2\lambda^4\mu^2 + C_4\lambda^4\mu + C_5\lambda^8 = 0.$$

This is of the form

$$(\mu^2+\epsilon_1\mu+\delta_1\lambda^4)(\mu^2+\epsilon_2\mu+\delta_2\lambda^4)=0,$$

and in the limit we have

$$\left.\begin{aligned}\epsilon_1&=\frac{1}{2}\left(C_1-\frac{N}{K}\right)\\ \epsilon_2&=\frac{1}{2}\left(C_1+\frac{N}{K}\right)\\ \delta_1&=\frac{C_2+K}{2}\\ \delta_2&=\frac{C_2-K}{2}\end{aligned}\right\},$$

where $$K^2=C_2^2-4C_5$$

$$N=2C_4-C_1C_2.$$

Thus $$\mu=-\frac{\epsilon_1}{2}\pm\sqrt{\delta_1}\lambda^2 i,\quad -\frac{\epsilon_2}{2}\pm\sqrt{\delta_2}\lambda^2 i.$$

The conditions for stability in these modes are :

$$C_4>0$$

and $$X=\tfrac{1}{4}(C_1^2K^2-N^2)>0.$$

Violation of either of these conditions leads to a positive value of either ϵ_1 or ϵ_2.

Hence the higher modes have constant positive or negative damping of amount $-\frac{1}{2}\left(C_1\pm\frac{N}{K}\right)$.

Their frequencies are $\frac{\sqrt{C_2\mp K}}{2\sqrt{2}\pi}\lambda^2$,

where $\lambda=(n+\frac{1}{2})\pi$, and n is large.

11. *The Motion* in vacuo.

When the structure vibrates *in vacuo* its energy is constant and the modes are undamped oscillations. The analysis in this case admits of a concise summary.

The coefficients C_1, C_3, C_4, C_6 vanish. Thus the frequency equation is

$$\left(\frac{\mu^2}{\lambda^4}\right)^2+C_2\left(\frac{\mu^2}{\lambda^4}\right)+C_5=0.$$

This, together with

$$\cosh r\lambda\cos r\lambda+1=0,$$

determines the modes.

The two solutions associated with any value λ_1 satisfying the transcendental equation are:

$$\lambda_1, \quad \mu^2 = -e^2\lambda_1^4, \quad \lambda_2^4 = p^4\lambda_1^4$$

and

$$\lambda_1, \quad \mu^2 = -f^2\lambda_1^4, \quad \lambda_2^4 = q^4\lambda_1^4,$$

where

$$e^2 = \tfrac{1}{2}(C_2 - K), \quad p^4 = \frac{C_2(C_2 - K)}{2C_5} - 1$$

$$f^2 = \tfrac{1}{2}(C_2 + K), \quad q^4 = \frac{C_2(C_2 + K)}{2C_5} - 1,$$

and K has been defined in § 10.

Thus the frequencies are proportional to $e\lambda_1^2$ and $f\lambda_1^2$, and the modes deriving from a single value of λ_1 are:

$$\begin{aligned} &A_1 \cosh \lambda_1 x \cos (e\lambda_1^2 t + \epsilon_1) + A_2 \sinh \lambda_1 x \cos (e\lambda_1^2 t + \epsilon_2) \\ &\quad + A_3 \cos \lambda_1 x \cos (e\lambda_1^2 t + \epsilon_3) A_4 \sin \lambda_1 x \cos (e\lambda_1^2 t + \epsilon_4) \\ &+ A_5 \cosh p\lambda_1 x \cos (e\lambda_1^2 t + \epsilon_5) + A_6 \sinh p\lambda_1 x \cos (e\lambda_1^2 t + \epsilon_6) \\ &\quad + A_7 \cos p\lambda_1 x \cos (e\lambda_1^2 t + \epsilon_7) + A_8 \sin p\lambda_1 x \cos (e\lambda_1^2 t + \epsilon_8), \end{aligned}$$

and a second series in which f is substituted for e and q for p.

If the mass of the system is wholly concentrated in the spars, the frequencies can be simply expressed in terms of characteristic points of a cross-section. P_1 and P_2 being the spar centres, let G be the centre of mass of the section, and F the centroid of B_1 and B_2 the flexural rigidities of the spars, so that F may be called the "centre of flexure" of the two spars. Then if $S = B_1 + B_2$ and M is the mass of the section, the frequencies are proportional to

$$\lambda_1^2 \sqrt{\frac{FP_1}{GP_1} \cdot \frac{S}{M}} \quad \text{and} \quad \lambda_1^2 \sqrt{\frac{FP_2}{GP_2} \cdot \frac{S}{M}}.$$

If, in addition, the spars are of the same shape and material, the frequencies are proportional to

$$\lambda_1^2 \sqrt{\frac{B_1}{M_1}} \quad \text{and} \quad \lambda_1^2 \sqrt{\frac{GP_1}{GP_2} \cdot \frac{B_1}{M_1}},$$

where B_1, M_1 refer to the spar P_1.

12. I have to thank the Air Ministry for permission to publish this work, which in its original form was a contribution to the theory of wing flutter. It should be emphasized that all statements and conclusions which it contains are advanced on my sole responsibility, and carry no official sanction.

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IX. *On the Relations between the Fourier Constants of a Periodic Function and the Coefficients determined by Harmonic Analysis.* By ALBERT EAGLE, *B.Sc., Lecturer in Mathematics in the Victoria University of Manchester* *.

IF $f(t)$ is a periodic function, of period 2π, the coefficients of the nth cosine and sine harmonics are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

respectively. These coefficients are termed the "Fourier constants" of the function.

If $f(t)$ is a function for which these integrals cannot be evaluated, or if it is a function which is only known empirically, these integrals—in the absence of some mechanical harmonic analyser—can only be evaluated approximately by summing the value of the integrand at equidistant ordinates. This constitutes the ordinary method of harmonic analysis; and it can be shown that if we take $2p$ ordinates per period at intervals of π/p, and if A_q denote the value of $f(t)$ at $t=q\pi/p$, the curve given by

$$y = \frac{\mathrm{a}_0}{2} + \mathrm{a}_1\cos t + \ldots + \mathrm{a}_{p-1}\cos \overline{p-1}\,t + \frac{\mathrm{a}_p}{2}\cos pt$$
$$+ \mathrm{b}_1\sin t + \ldots + \mathrm{b}_{p-1}\sin \overline{p-1}\,t, \quad . \quad . \quad . \quad . \quad (2)$$

where

$$\mathrm{a}_n = \frac{2}{p}\Sigma \mathrm{A}_q \cos qn\pi/p \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

and

$$\mathrm{b}_n = \frac{2}{p}\Sigma \mathrm{A}_q \sin qn\pi/p \; \dagger, \quad . \quad . \quad . \quad . \quad (2b)$$

will pass through tops of the given ordinates.

Now, the value of these coefficients, as thus determined, can be expressed in terms of the Fourier constants by the equations

$$\mathrm{a}_n = a_n + (a_{2p-n} + a_{2p+n}) + (a_{4p-n} - a_{4p+n}) + \ldots \quad (3)$$

and

$$\mathrm{b}_n = b_n - (b_{2p-n} - b_{2p+n}) - (b_{4p-n} - b_{4p+n}) - \ddagger \ldots \quad (4)$$

* Communicated by the Author.

† A. Eagle, 'A Practical Treatise on Fourier's Theorem and Harmonic Analysis,' p. 77 (Longmans, 1925).

‡ *Loc. cit* p. 97.

Further, if the periodic function is an artificial one built up of arcs of parabolas of different degrees, the Fourier constants may be expressed in terms of the discontinuities of the function, and of its several differential coefficients, at the points where the different parabolic arcs join one another. It will be convenient to call a discontinuity that is a sudden jump in magnitude of the rth differential coefficient, a discontinuity of the rth order, and to denote it by I^r, so that $\mathrm{I}^r \equiv f^r(t+0) - f^r(t-0)$ at the point of discontinuity. That being so, it can be shown that if, in one period, the function has discontinuities $\mathrm{I}_1, \mathrm{I}_2, \ldots$ of zero order; $\mathrm{I}_1', \mathrm{I}_2', \ldots$ of the first order; $\mathrm{I}_1'', \mathrm{I}_2'', \ldots$ of the second order, and so on, and if $t=\alpha$ is the location of any of these points of discontinuity, then

$$\pi a_n = -\frac{1}{n}\Sigma \mathrm{I} \sin n\alpha - \frac{1}{n^2}\Sigma \mathrm{I}' \cos n\alpha$$

$$+\frac{1}{n^3}\Sigma \mathrm{I}'' \sin n\alpha + \frac{1}{n^4}\Sigma \mathrm{I}''' \cos n\alpha \ldots \quad . \quad (5)$$

and

$$\pi b_n = \frac{1}{n}\Sigma \mathrm{I} \cos n\alpha - \frac{1}{n^2}\Sigma \mathrm{I}' \sin n\alpha$$

$$-\frac{1}{n^3}\Sigma \mathrm{I}'' \cos n\alpha + \frac{1}{n^4}\Sigma \mathrm{I}''' \sin n\alpha + \ldots, \quad . \quad (6)$$

where in each series two positive terms and two negative terms alternate *. The series terminate if the parabolic arcs are all of finite degree, but they may often be used when the arcs are parabolic arcs of an infinite degree such as arcs of sine or exponential curves etc.

The formulæ in (5) and (6) show clearly that *the higher harmonics ultimately depend only upon the discontinuities of lowest order that are present*, while if the function only possesses discontinuites of a single order, the formulæ reduce to a single term.

Let us examine this important case in further detail, and suppose that the function is also analysed by the ordinary method of harmonic analysis, and also that the number of ordinates per half period, p, is chosen *so that all the discontinuities lie on the measured ordinates*; that is α, in (5) and (6), is of the form $\mathrm{N}\pi/p$, where N is an integer. Let us now substitute these expressions for the a's and b's in (3) and (4). Since $(2p \pm n)\alpha = (2p \pm n)\mathrm{N}\pi/p = 2\pi\mathrm{N} \pm n\alpha$ and

* *Loc. cit.* p. 59.

so on, we get, in the case where all the discontinuities are of zero order, so that

$$\pi a_n = -\frac{1}{n}\Sigma \mathrm{I}\sin n\alpha \quad \text{and} \quad \pi b_n = \frac{1}{n}\Sigma \mathrm{I}\cos n\alpha,$$

$$\pi \mathrm{a}_n = -\frac{1}{n}\Sigma \mathrm{I}\sin n\alpha + \frac{1}{2p-n}\Sigma \mathrm{I}\sin n\alpha + \frac{1}{2p+n}\Sigma \mathrm{I}\sin n\alpha + \ldots\ldots$$

and

$$\pi \mathrm{b}_n = \frac{1}{n}\Sigma \mathrm{I}\cos n\alpha - \frac{1}{2p-n}\Sigma \mathrm{I}\cos n\alpha + \frac{1}{2p+n}\Sigma \mathrm{I}\cos n\alpha + \ldots\ldots \quad (7)$$

If we divide these expressions, respectively, by πa_n and πb_n, they both give

$$\frac{\mathrm{a}_n}{a_n} = \frac{\mathrm{b}_n}{b_n} = n\left\{\frac{1}{n} - \frac{1}{2p-n} + \frac{1}{2p+n} - \frac{1}{4p-n} + \frac{1}{4p+n} + \ldots\right\}. \quad (8')$$

Now, the series in brackets is the function $\frac{\pi}{2p}\cot\frac{\pi n}{2p}$, whence we have

$$\frac{a_n}{\mathrm{a}_n} = \frac{b_n}{\mathrm{b}_n} = \frac{\tan\frac{\pi n}{2p}}{\frac{\pi n}{2p}}. \quad \ldots\ldots \quad (8)$$

Hence for a periodic function which only possesses discontinuities of a zero order, and which is necessarily a step-like function consisting of portions of straight lines parallel to the axis of t together with portions of the measured ordinates joining them, we see that *we can determine the true Fourier constants by merely calculating the* a*'s and* b*'s* by (2*a*) and (2*b*) * and multiplying the result by the function in (8). Moreover, this result, which is exact under the conditions specified, tends to become true for large values of n for *all functions* which possess discontinuities of zero order situated on the measured ordinates.

If next we suppose the function only to possess discontinuities of the first order, so that it necessarily consists of

* $\frac{1}{2}f(t+0)+\frac{1}{2}f(t-0)$ is to be taken as the value of the ordinates in calculating these formulæ.

joined portions of straight lines, all of finite slope, and if all the junctions are on the measured ordinates, we have

$$\pi a_n = -\frac{1}{n^2}\Sigma\mathrm{I}' \cos n\alpha \quad \text{and} \quad \pi b_n = -\frac{1}{n^2}\Sigma\mathrm{I}' \sin n\alpha.$$

Substituting these results in (3) and (4) as before, and dividing by a_n and b_n respectively, it will be found that both give

$$\frac{\mathrm{a}_n}{a_n} = \frac{\mathrm{b}_n}{b_n} = n^2\left\{\frac{1}{n^2} + \frac{1}{(2p-n)^2} + \frac{1}{(2p+n)^2} + \dots\right\}. \quad (9')$$

Now, this series in brackets is the function $\frac{\pi^2}{4p^2}\operatorname{cosec}^2\frac{\pi n}{2p}$, whence

$$\frac{a_n}{\mathrm{a}_n} = \frac{b_n}{\mathrm{b}_n} = \frac{\sin^2\frac{\pi n}{2p}}{\left(\frac{\pi n}{2p}\right)^2}. \quad (9)$$

Exactly similarly, if $f(t)$ only possesses discontinuities of the second order, all situated on the measured ordinates, so that it necessarily consists of parabolic arcs joined without any discontinuity of slope, we find

$$\frac{\mathrm{a}_n}{a_n} = \frac{\mathrm{b}_n}{b_n} = n^3\left\{\frac{1}{n^3} - \frac{1}{(2p-n)^3} + \frac{1}{(2p+n)^3} - \dots\right\}, \quad (10')$$

which is readily identified as being

$$\frac{a_n}{\mathrm{a}_n} = \frac{b_n}{\mathrm{b}_n} = \frac{\sin^3\frac{\pi n}{2p}}{\left(\frac{\pi n}{2p}\right)^3 \cos\frac{\pi n}{2p}}. \quad (10)$$

Lastly, for our present examples, if the discontinuities (all situated on the measured ordinates) are all of the third order, so that $f(t)$ consists of parabolic arcs of the third degree joined together without any sudden breaks either of slope or curvature, it will be found that

$$\frac{\mathrm{a}_n}{a_u} = \frac{\mathrm{b}_u}{b_n} = n^4\left\{\frac{1}{n^4} + \frac{1}{(2p-n)^4} + \frac{1}{(2p+n)^4} + \dots\right\}, \quad (11')$$

giving

$$\frac{a_n}{\mathrm{a}_n} = \frac{b_u}{\mathrm{b}_n} = \frac{3\sin^4\frac{\pi n}{2p}}{\left(\frac{\pi n}{2p}\right)^4\left\{1 + 2\cos^2\frac{\pi n}{2p}\right\}}\,{}^*. \quad (11)$$

* This result is easily obtained by observing that the series (11′) is the derivative, with respect to n, of (10′), and that in turn is the derivative of (9′).

For readiness in use these four functions in (8)–(11), which we will denote by F_0, F_1, F_2, and F_3 respectively, are tabulated below, from which table graphs of the functions, sufficiently accurate for all ordinary use, may readily be constructed. In the case of the functions F_0 and F_2 it will obviously be best to graph the reciprocals, which are readily obtainable by a slide-rule.

TABLE I.

$12n/p$.	$F_0(n/p)$.	$F_1(n/p)$.	$F_2(n/p)$.	$F_3(n/p)$.
0	1·000	1·000	1·000	1·000
1	1·006	·995	1·000	1·000
2	1·024	·978	1·000	1·000
3	1·055	·949	1·002	·999
4	1·103	·912	1·006	·998
5	1·172	·865	1·014	·994
6	1·273	·810	1·032	·985
7	1·422	·750	1·067	·969
8	1·654	·684	1·131	·935
9	2·049	·615	1·260	·877
10	2·852	·545	1·553	·785
11	5·275	·474	2·500	·652
12	∞	·405	∞	·493

If the discontinuities, while all of a single order, are all of them *situated midway between the measured ordinates*, it is easily seen that the effect will be to change the sign of alternate terms in the series in (8′), (9′), (10′), and (11′) so that in place of the four F-functions we shall now have series representing the reciprocals of

$$\frac{\sin\frac{n\pi}{2p}}{\frac{n\pi}{2p}}, \qquad \left(\frac{\sin^2\frac{n\pi}{2p}}{\frac{n\pi}{2p}}\right)^2 \sec\frac{n\pi}{2p},$$

$$\left(\frac{\sin\frac{n\pi}{2p}}{\frac{n\pi}{2p}}\right)^3 \cdot \frac{2}{1+\cos^2\frac{n\pi}{2p}} \quad \text{and} \quad \left(\frac{\sin\frac{n\pi}{2p}}{\frac{n\pi}{2p}}\right)^4 \frac{6\sec\frac{n\pi}{2p}}{5+\cos^2\frac{n\pi}{2p}}$$

respectively.

These four functions, denoted respectively by M_0, M_1, M, and M_3, are tabulated in Table II.

Table II.

$12n/p$.	$M_0(n/p)$.	$M_1(n/p)$.	$M_2(n/p)$.	$M_3(n/p)$.
0.........	1·000	1·000	1·000	1·000
1.........	·997	1·003	1·000	1·000
2.........	·987	1·012	1·000	1·000
3.........	·974	1·028	·998	1·001
4.........	·955	1·053	·995	1·002
5.........	·930	1·090	·987	1·005
6.........	·900	1·146	·973	1·014
7.........	·866	1·232	·947	1·032
8.........	·827	1·368	·905	1·069
9.........	·784	1·607	·841	1·152
10.........	·738	2·110	·754	1·355
11.........	·688	3·632	·643	2·059
12.........	·637	∞	·516	∞

The F_0, M_1, F_2, and M_3 functions are infinite when $n=p$ because, in the cases of even-order discontinuities *at* the ordinates or odd-order ones *midway between*, it is possible to have a non-zero periodic function, of half period equal to the distance between the ordinates, and of arbitrary amplitude, when all the ordinates are zero. With first-order discontinuities this function is saw-edge-like and the other cases are also easily seen. Hence the a's and b's will be finite when $n=p$, while the a's and b's are zero.

The fact that these F and M functions enable the integrals

$$\frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt \quad \text{and} \quad \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt$$

to be evaluated exactly, in the special conditions assumed, by multiplying the summation results given by ($2a$) and ($2b$) by these functions, is truly remarkable and deserves reflecting upon.

As an illustration of the utility of these correction factors we will take an example worked out in my book * of an odd periodic function of period 2π obtained by joining the point $(\frac{2}{5}\pi, 24)$ to the origin and to the point $(\pi, 0)$ by straight lines. The exact harmonic analysis of this function, to two decimal places, is

$$\begin{aligned} f(t) = {} & 19{\cdot}27\sin t + 2{\cdot}98\sin 2t - 1{\cdot}32\sin 3t \\ & - 1{\cdot}20\sin 4t + {\cdot}53\sin 6t + {\cdot}24\sin 7t \\ & - {\cdot}18\sin 8t - {\cdot}24\sin 9t + \ldots . \end{aligned}$$

* *Loc. cit.* pp. 90–91.

When analysed with 10 ordinates per half-period, the result was

$$f(t)=19{\cdot}43 \sin t+3{\cdot}08 \sin 2t-1{\cdot}43 \sin 3t$$
$$-1{\cdot}38 \sin 4t+{\cdot}73 \sin 6t+{\cdot}37 \sin 7t$$
$$-{\cdot}33 \sin 8t-{\cdot}49 \sin 9t.$$

Multiplying these coefficients by the appropriate values of the function $F_1(n/p)$ brings these two results into agreement.

Even when a periodic function possesses discontinuities of different orders, it will generally be better to correct all the a's and b's by the function appropriate to the lowest-order discontinuities than to leave them uncorrected. As an illustration we will take the practical case of a rectified sine-waved voltage in which a constant voltage is kept back by the polarizing effect of the rectifying device. Taking, for the purpose of illustration, the polarization voltage as half the maximum voltage, and making the period 2π, we have $f(t)$ an even function given by $f(t)=2 \cos\frac{t}{2}-1$ from $t=0$ to $t=\frac{2\pi}{3}$, and $f(t)=0$ from $t=\frac{2\pi}{3}$ to $t=\pi$.

The exact harmonic analysis of this function is readily found to be

$$f(t)=0{\cdot}4363+{\cdot}5515 \cos t+{\cdot}0551 \cos 2t$$
$$-{\cdot}0630 \cos 3t+{\cdot}0197 \cos 4t+{\cdot}0100 \cos 5t$$
$$-{\cdot}0154 \cos 6t+\ldots;$$

while the ordinary harmonic analysis, with six ordinates per half period, is found to give

$$f(t)=0{\cdot}4297+{\cdot}5577 \cos t+{\cdot}0619 \cos 2t$$
$$-{\cdot}0773 \cos 3t+{\cdot}0275 \cos 4t+{\cdot}0197 \cos 5t$$
$$-{\cdot}0185 \cos 6t.$$

If these coefficients be multiplied by the values of the function $F_1(n/p)$ for values of $12n/p$ of 0, 2, 4, 6, 8, 10, and 12 respectively, and if the last coefficient is then doubled to make up for the fact that this coefficient is, in accordance with equation (2), $\frac{1}{2}a_6$ and not a_6, we get

$$f(t)=0{\cdot}4297+{\cdot}5454 \cos t+{\cdot}0564 \cos 2t$$
$$-{\cdot}0626 \cos 3t+{\cdot}0188 \cos 4t+{\cdot}0107 \cos 5t$$
$$-{\cdot}0147 \cos 6t+\ldots,$$

which is almost indistinguishable from the exact harmonic analysis if allowance is made for the somewhat large errors in the first two terms—which is a *special effect* due to the

very few ordinates per half period combined with the fact that the convexity is always of the same sign—although the function possesses discontinuities of all orders save zero.

Application to Empirical Functions.

The ordinary method of harmonic analysis, when applied to ordinary experimental curves, such as oscillograms of voltage wave forms, has very similar disadvantages of lack of accuracy as it has in the case of artificial functions, which we have seen how to correct above. In order to overcome this I have, in my book *, given a more accurate method. Two varieties of this method consist, respectively, in drawing parabolic arcs from the tops of one even-numbered ordinate to the next, passing through the top of the intervening ordinate, then drawing a similar succession of arcs starting and terminating at the odd numbered ordinates, and passing through the even ones, and taking the mean of these two results for the first case; and in the second case, in drawing third-degree parabolic arcs from the top of one ordinate to the top of the next, and making them parallel, at their extremities, to the lines joining the tops of the two ordinates adjacent to the end ordinates, and taking this as the given function. In either case the exact analysis of these functions is obtained; and in both cases it appears that both the second and fourth differences of the ordinates are required, when the analysis can be effected by using these differences instead of the ordinates in the ordinary Schedules for harmonic analysis constructed for the convenient carrying out of the formulæ in (2a) and (2b).

I wish now to point out the fact that the result of this method can be obtained much more readily by merely multiplying the result obtained by the ordinary method of harmonic analysis by functions similar to the F and M functions tabulated above.

Since in *all* methods of harmonic analysis the results are linear functions of the $2p$ ordinates, the comparison of different methods resolves itself into a comparison of the different functions which we take to represent the case in which all the ordinates are zero, except one, which is unity, and which we will imagine is situated at $t=0$. The ordinary method represents this function by

$$f(t)=\frac{1}{p}\left\{\frac{1}{2}+\cos t+\cos 2t+\ldots+\cos\overline{p-1}\,t+\frac{1}{2}\cos pt\right\}; \quad \ldots\ldots (12)$$

* *Loc. cit.* pp. 102–111.

that is, by

$$\frac{\sin pt}{p \sin t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12\,a)$$

To compare different functions we will take the ordinates at unit distances of x apart and the non-zero ordinate as situated at $x=0$. If the function passing through these points becomes negligible before $x=p$, we may extend the zero ordinates to $\pm\infty$. This set of points, viz. (0, 1) and the points $y=0$ for all other positive and negative integral values of x, we will call the "Fundamental Set of Points," and the spaces between adjacent ordinates "intervals"—the two between -1 and $+1$ being called "central intervals" and all the others "ordinary intervals."

From (12a) we see, on replacing t by $\pi x/p$, that the ordinary method of harmonic analysis adopts

$$\frac{\sin \pi x}{p \sin \pi x/p}, \approx \frac{\sin \pi x}{\pi x} \quad . \quad . \quad . \quad . \quad . \qquad (12\,b)$$

if p is large, as the function through the fundamental set of points. This function suffers from the disadvantage of giving far too much disturbance for moderately large values of x. Little can be said in defence of the function making the displacement at the centre of the second ordinary interval only 3/5 of that at the centre of the first one, while it makes the displacement at the centre of the tenth ordinary interval 19/21 of that in the ninth.

We will now give some artificial functions passing through this fundamental set of points. In giving the equations of the different arcs of which these functions are built up, we shall in all cases suppose each arc is referred to an origin at its left-hand extremity so that x will always go from 0 to 1 for each arc.

It will easily be found that the first variety of my method adopts the function : $y=1-\frac{3}{4}x-\frac{1}{4}x^2$ for the central interval ; $y=-\frac{1}{4}x(1-x)$ for the first ordinary interval, and zero beyond ; while the second variety takes, $y=1-\frac{5}{2}x^2+\frac{3}{2}x^3$ for the central interval ; $y=-\frac{1}{2}x(1-x)^2$ for the first ordinary interval, and zero beyond.

The harmonic analysis of the first of these functions, when the ordinates are reduced to π/p apart, is found to give, instead of the result in (12),

$$f(t)=\frac{1}{p}\left\{\frac{1}{2}+c_1 \cos t+c_2 \cos 2t+\ldots\right\}, \quad . \qquad (13)$$

where

$$c_n = \left(\frac{\sin\frac{n\pi}{2p}}{\frac{np}{2p}}\right)^3 \left\{\frac{n\pi}{2p}\sin\frac{n\pi}{2p} + \cos\frac{n\pi}{2p}\right\}; \quad . \quad . \quad (13a)$$

while the harmonic analysis of the second function is likewise found to give the same series, but with

$$c_n = \left(\frac{\sin\frac{n\pi}{2p}}{\frac{n\pi}{2p}}\right)^3 \left\{\frac{6p}{n\pi}\sin\frac{n\pi}{2p} - 2\cos\frac{n\pi}{2p}\right\} \quad . \quad . \quad (13b)$$

These results are easily deducible from (5), using the discontinuities between the different arcs.

If we had taken the non-zero ordinate at any other position than $t=0$, (12) would have involved both sine and cosine harmonics, and so in that case would (13) likewise; but it is obvious that the ratio of both the nth sine and the nth cosine harmonic in (13) to the corresponding terms in (12) would still be c_n in both cases. Since any function to be analysed consists of a sum of these single non-zero-ordinate functions, we see that in all cases the new method will give both the nth harmonic terms, c_n times the magnitude given by the ordinary method.

Indeed, it is obvious that *every possible method of harmonic analysis can only give the same result as the ordinary method with both the* n*th sine and* n*th cosine harmonics multiplied by some function of n/p.*

Very much better functions through the fundamental set of points may be obtained than any of the three already given; for it is possible to join arcs of third-degree parabolas without any discontinuities either of slope or curvature. In fact, we know from the theory of beams, that a uniform straight flexible lath made to pass through the points automatically gives us this function, provided that the horizontal scale of the function is sufficiently large so that the inclination of the lath to the t axis is everywhere small.

To obtain the equation of this interesting function, which we will call the "lath function," we note that the equation of the arc, in the (right hand) central interval, must be of the form

$$y = 1 - cx^2 + (c-1)x^3; \quad . \quad . \quad . \quad . \quad (14)$$

also, that if p_n and $2q_n$ denote the values at the end of the first and second differential coefficients at the beginning of

any ordinary interval, and $-p_{n+1}$ and $-2q_{n+1}$ the values at the end of that interval (and the beginning of the next), we have

$$y=p_nx+q_nx^2-(p_n+q_n)x^3$$

for the equation in the interval concerned ; whence

$$p_{n+1}=2p_n+q_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and

$$q_{n+1}=3p_n+2q_n ;$$

whence both the p's and q's satisfy the equation

$$u_{n+2}-4u_{n+1}+u_n=0 ;$$

and hence

$$p_n=\mathrm{A}\mathrm{Q}^n+\mathrm{A}'\mathrm{Q}^{-n}$$

and

$$q_n=\mathrm{B}\mathrm{Q}^n+\mathrm{B}'\mathrm{Q}^{-n},$$

where $\mathrm{Q}=2-\sqrt{3}$ and $\mathrm{Q}^{-1}=2+\sqrt{3}$. If the p's and q's $\rightarrow 0$ as $n\rightarrow\infty$, we must have $\mathrm{A}'=\mathrm{B}'=0$, when substitution in (15) shows that $\mathrm{B}=-\mathrm{A}\sqrt{3}$; so that the equation of the arc in any ordinary interval is given by

$$y=\mathrm{A}\{x-\sqrt{3}x^2+(\sqrt{3}-1)x^3\}.$$

Determining A and c so that this shall fit on to (14) without any discontinuities of slope or curvature, we find

$$c=3\sqrt{3}-3 \quad \text{and} \quad \mathrm{A}=3\sqrt{3}-6 ;$$

so that

$$y=1-(3\sqrt{3}-3)x^2+(3\sqrt{3}-4)x^3$$

in the (right hand) central interval and

$$y=(3\sqrt{3}-6)x+(6\sqrt{3}-9)x^2+(15-9\sqrt{3})x^3$$

in the first ordinary interval; and in successive ordinary intervals we have merely to multiply this equation by successive powers of $-(2-\sqrt{3})$.

This function, at the middle of the fifth ordinary interval, has a value of only 1/1500, whereas the function $(\sin\pi x)/\pi x$ has then a value of $2/11\pi$—nearly 90 times as great: we may note that it decays exponentially at the same rate as $e^{x\log(2-\sqrt{3})}=e^{-1\cdot 31x}$ approx.

We must now obtain the harmonic analysis of this lath-function, which is easily done by considering the discontinuities. These are all of the third order and are merely 1/6 of the difference of the coefficients of x^3 in successive intervals. Hence we have a discontinuity of $(6\sqrt{3}-8)/6$ at $x=0$; one of $(19-12\sqrt{3})/6$ at $x=1$; one of $-12+7\sqrt{3}$ at $x=2$; and thence after discontinuities

equal to this last multiplied by successive powers of $-(2-\sqrt{3})$. Using (5) to determine the Fourier constants, and making use of the formula

$$1+2c\cos\theta+2c^2\cos 2\theta+2c^2\cos 3\theta+\ldots$$
$$=\frac{1-c^2}{1-2c\cos\theta+c^2},$$

we easily find—remembering that when the ordinates are reduced from unit distance apart to π/p all the discontinuities, being of the third order, will be increased in the ratio $(p/\pi)^3$—that the harmonic analysis is given by (13), where

$$c_n=\left(\frac{\sin\dfrac{n\pi}{2p}}{\dfrac{n\pi}{2p}}\right)^4\left\{\frac{3}{2+\cos n\pi/p}\right\}. \quad . \quad . \quad (16)$$

It will be noticed that this expression is the same as that in (11), which it should be, since both refer to periodic functions which only have third-order discontinuities and have them on the measured ordinates.

We leave it as an interesting rider to the reader to prove that this lath function, from $x=1$ to $x=\infty$, is proportional to

$$(2-\sqrt{3})^x\{a_1\cos\pi x+a_3\cos 3\pi x+a_5\cos 5\pi x+\ldots$$
$$+b_1\sin\pi x+b_3\sin 3\pi x+b_5\sin 5\pi x+\ldots\},$$

where

$$a_n=\frac{n^4-6n^2c^2+c^4}{(n^2+c^2)^4} \quad\text{and}\quad b_n=\frac{4(cn^3-c^3n)}{(n^2+c^2)^4},$$

and where $\pi c=\log_e(2+\sqrt{3})$,

i. e. $c=0{\cdot}41920\ldots$ and $c^2=0{\cdot}17573\ldots$

A simpler artificial function than the lath-function can be constructed through the fundamental set of points out of ordinary parabolic arcs by joining them, without any discontinuity of slope, *midway between the measured ordinates.*

The equation of the central arc must be of the form

$$y=1-cx^2 \quad\text{for}\quad -\tfrac{1}{2}<x<\tfrac{1}{2}\,; \quad . \quad . \quad . \quad (17)$$

while if r_n and s_n are the values of the ordinate and slope respectively at the beginning of any other arc, and $-r_{n+1}$ $-s_{n+1}$ the values of these quantities at the end of the same arc, so that the equation of the arc—remembering that it must make $y=0$ at its mid point—is

$$y=r_n+s_nx-(4r_n+2s_n)x^2,$$

we easily find

$$r_{n+1}=3r_n+s_n \quad \ldots \ldots \quad (18)$$

and $$s_{n+1}=8r_n+3s_n\,;$$

whence the r's and s's satisfy the equation

$$u_{n+2}-6u_{n+1}+u_n=0\,;$$

and so

$$r_n=\mathrm{A}\mathrm{Q}^n+\mathrm{A}'\mathrm{Q}^{-n} \quad\text{and}\quad s_n=\mathrm{B}\mathrm{Q}^n+\mathrm{B}'\mathrm{Q}^{-n}$$

where $$\mathrm{Q}=3-\sqrt{8} \quad\text{and}\quad \mathrm{Q}^{-1}=3+\sqrt{8}.$$

If the r's and s's $\to 0$ as $n\to\infty$, we must have $\mathrm{A}'=\mathrm{B}'=0$, and then substitution in (18) shows that $\mathrm{B}=-\sqrt{8}\mathrm{A}$, so that the equation of the first ordinary arc is of the form

$$y=\mathrm{A}\left\{1-\sqrt{8}x+(2\sqrt{8}-4)x^2\right\}.$$

Making this, at $x=0$, join without discontinuity of slope to $y=1-cx^2$ at $x=\frac{1}{2}$, we get

$$\mathrm{A}=1-c/4 \quad\text{and}\quad -\mathrm{A}\sqrt{8}=-c,$$

whence $$y=1-4(\sqrt{2}-1)x^2$$

is the equation of the central arc, and

$$y=2-\sqrt{2}-(4\sqrt{2}-4)x+(12\sqrt{2}-16)x^2$$

is the equation of the first ordinary arc, and successive ordinary arcs are given by multiplying this equation by successive powers of $-(3-\sqrt{8})$. This function decays very rapidly; approximately as $e^{-1\cdot 76x}$, and at the middle of the fourth ordinary interval has a value of only about 1/2000.

The harmonic analysis of this function, which we will call the "parabola-function," is similarly found to be the series (13) with

$$c_n=\left(\frac{\sin\frac{n\pi}{2p}}{\frac{n\pi}{2p}}\right)^3\left\{\frac{4}{3+\cos n\pi/p}\right\}. \quad \ldots \quad (19)$$

This expression, it will be noted, is the same as the M_2 function of Table II., which again is as it should be.

Various analytic functions may be found to pass through the fundamental set of points. Three, which immediately suggest themselves, are

$$g\frac{\sin\pi x}{\sinh g\pi x}, \quad \frac{\sin\pi x}{\pi x\cosh h\pi x} \quad\text{and}\quad e^{-k^2\pi^2x^2}\frac{\sin\pi x}{\pi x},$$

which we will denote by $\phi(x)$, $\phi_h(x)$, and $\phi_k(x)$ respectively. For these to give at all reasonably "smooth" curves

through the points, g, h, and k must be small. A good way of comparing the suitability of the different functions is to tabulate the curvature they give at $x=0$ and the numerical slope they give at $x=1$; the results are :—

Function.	Curvature at $x=0$.	Slope at $x=1$.
$\dfrac{\sin \pi x}{\pi x}$	$\pi^2/3$	1
$\phi_g(x)$	$\pi^2(1+g^2)/3$	$1-g^2\pi^2/6$
$\phi_h(x)$	$\pi^2(1+3h^2)/3$	$1-h^2\pi^2/2$
$\phi_k(x)$	$\pi^2(1+6k^2)/3$	$1-k^2\pi^2$
Lath-fn.	4·392	0·804
Parabola-fn.	3·314	0·686

Looking at the graphs, my eyes prefer a slope of between 0·8 and 0·9. The numerical values adopted in Table III. B are $g^2=0{\cdot}12$, $h^2=0{\cdot}04$, and $k^2=0{\cdot}02$, giving a slope of practically 0·8 in all three cases.

There is, however, another vital condition which must be satisfied before any of these functions can be entertained, and which the lath and parabola functions automatically obey; that is, that if all the ordinates of the periodic function to be analysed are unity, we must have $y=1$ as the curve through the tops of the ordinates. This requires that

$$\ldots+\phi(x-1)+\phi(x)+\phi(x+1)+\phi(x+2)+\ldots \quad . \quad (20)$$

should be unity for all values of x. But this expression is a periodic function, P (x) say, of period unity given by

$$\mathrm{P}(x)=\tfrac{1}{2}\alpha_0+\alpha_1\cos 2\pi x+\alpha_2\cos 4\pi x+\ldots, \quad . \quad (21)$$

where α_m is given by

$$\alpha_m=4\int_0^{\infty}\phi(x)\cos 2m\pi x\,dx\,*, \quad . \quad . \quad (21\,a)$$

$\phi(x)$ being an even function. Hence we may take for $\phi(x)$ any function which passes through the fundamental set of points, provided we divide it by the expression in (21).

Applying this test to the three chosen functions in succession, we have

$$\alpha_m=4g\int_0^{\infty}\frac{\sin \pi x}{\sinh g\pi x}\cos 2m\pi x\,dx$$

$$=\frac{2g}{\pi}\int_0^{\infty}\frac{\sin\,(2m+1)x-\sin\,(2m-1)x}{\sinh gx}dx \quad . \quad . \quad (22')$$

$$=\tanh\frac{(2m+1)\pi}{2g}-\tanh\frac{(2m-1)\pi}{2g} \quad . \quad . \quad . \quad . \quad (22)$$

in the first case;

* *Loc. cit.* p. 155.

$$\alpha_m = \frac{4}{\pi}\int_0^\infty \frac{\sin \pi x}{x \cosh h\pi x}\cos 2m\pi x \, dx$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin (2m+1)x - \sin (2m-1)x}{x \cosh hx} dx \quad . \quad . \quad (23')$$

$$= \frac{2}{\pi}\left\{\tan^{-1}\sinh\frac{(2m+1)\pi}{2h} - \tan^{-1}\sinh\frac{(2m-1)\pi}{2h}\right\} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

in the second case ; and

$$\alpha_m = \frac{4}{\pi}\int_0^\infty \frac{\sin \pi x}{x} e^{-k^2\pi^2x^2}\cos 2m\pi x \, dx$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin (2m+1)x - \sin (2m-1)x}{x} e^{-k^2x^2} dx \, . \quad (24')$$

$$= \mathrm{Erf}\left(\frac{2m+1}{2k}\right) - \mathrm{Erf}\left(\frac{2m-1}{2k}\right), \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where $\mathrm{Erf}(x)$ is the tabulated Error-function

$$\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2} dx,$$

(so that $\mathrm{Erf}(\infty)=1$) in the third case.

With the above values for g, h, and k it is easily seen that these values make $\frac{1}{2}\alpha_0$ unity to within about $10^{-3\cdot 64}$, $10^{-3\cdot 30}$, and $10^{-6\cdot 23}$ respectively ; make α_1 zero to within the same amounts, and make the higher α's differ from unity by incomparably smaller amounts. Hence for ordinary purposes any of these three functions may safely be adopted. We must now find the factors which these functions give for multiplying the ordinarily obtained harmonic coefficients. If we compress the ϕ_g function so that the zero ordinates are reduced to π/p apart, instead of unity, and suppose that p is not less than about 10 since the function is everywhere negligible past the ninth zero, we have its harmonic analysis given by (2), where

$$a_n = \frac{2}{\pi}\int_0^\pi \frac{g \sin pt}{\sinh gpt}\cos nt \, dt$$

$$= \frac{2g}{\pi p}\int_0^{p\pi} \frac{\sin t}{\sinh gt}\cos\frac{nt}{p} dt,$$

where the upper limit may now be replaced by infinity. But this expression is the same as that in (22′) multiplied by $1/2p$ with n/p replacing $2m$. Hence, by (13), the factor to be applied to the coefficients of the nth harmonic is

$$c_n = \frac{1}{2}\left\{\mathrm{than}\frac{(p+n)\pi}{2pg} + \tanh\frac{(p-n)\pi}{2pg}\right\}. \quad . \quad . \quad (26)$$

In an exactly similar manner ϕ_h and ϕ_k give

$$c_n = \frac{1}{\pi}\left\{\tan^{-1}\sinh\frac{(p+n)\pi}{2ph} + \tan^{-1}\sinh\frac{(p-n)\pi}{2ph}\right\} \quad . \quad (26)$$

and

$$c_n = \frac{1}{2}\left\{\operatorname{Erf}\left(\frac{p+n}{2kp}\right) + \operatorname{Erf}\left(\frac{p-n}{2kp}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

respectively.

The first term in each of (25), (26), and (27) is sensibly one-half, so that the expressions are odd functions of the ratio of $(1-n/p)$ to $g, h,$ or k, plus one-half. That is, if g, h, or k is increased in a given ratio, we shall get the same value for c_n, provided we increase $1-n/p$ in the same ratio. We may note that g, h, and k can never be large enough to render the first term appreciably different to $1/2$; for if they were given $\sqrt{2}$, the values we have taken for them, the slope of the curves at $x=1$ would be reduced from 0·8 to 0·6, which anyone's eye would reject for a smoothly-flowing curve through the fundamental set of points. These three functions for c_n are tabulated in Table III. B under the columns $\mathrm{A}g(n/p)$, $\mathrm{A}h(n/p)$, and $\mathrm{A}k(n/p)$ respectively.

An entirely different set of ϕ functions may be found in the following manner:—Denote α_m in (21a) by $\psi(m)$, then *

$$\phi(x) = \int_0^\infty \psi(m) \cos 2\pi x m \, dm.$$

If now, for simplicity, we take $\mathrm{P}(x)=1$, $\psi(m)$ must satisfy the following conditions:—$\psi(\mathrm{N})=0$ if N is an integer, $\psi(0)=2$, and

$$\int_0^\infty \psi(m) \cos 2\pi \mathrm{N} m \, dm = 0$$

if N is an integer and equals unity if $\mathrm{N}=0$. That

$$\left.\begin{aligned} &\psi(m) = 1 + \frac{\mathrm{H}(\frac{1}{2}-m)}{\mathrm{H}(\frac{1}{2})} \quad \text{for} \quad 0<m<1 \\ \text{and} \quad &\\ &\psi(m) = 0 \quad \text{for} \quad m>1, \end{aligned}\right\} \quad . \quad (28)$$

where H is any odd function, satisfies these condition will be evident on reflexion. Three interesting particular cases are:—$\psi(m)=2$ for $0<m<\frac{1}{2}$ and $\psi(m)=0$ for $m>\frac{1}{2}$; $\psi(m)=1+\cos\pi m$ for $0<m<1$ and $\psi(m)=0$ for $m>1$, which gives a second-order discontinuity at $m=1$; and

* *Loc. cit.* p 147.

$\psi(m)=1+\frac{9}{8}\cos\pi m-\frac{1}{8}\cos 3\pi m$ for $0<m<1$ and $\psi(m)=0$ for $m>1$, which give a fourth-order discontinuity at $m=1$.

The first case gives

$$\phi(x)=\int_0^{\frac{1}{2}} 2\cos 2\pi xt\,dt=\frac{\sin\pi x}{\pi x},$$

the function that ordinary harmonic analysis works on, while the second and third give

$$\phi(x)=\frac{\sin 2\pi x}{2\pi x}\cdot\frac{1}{1-4x^2}$$

and

$$\phi(x)=\frac{\sin 2\pi x}{2\pi x}\cdot\frac{1}{1-4x^2}\cdot\frac{9}{9-4x^2}$$

respectively.

There are, of course, other solutions for ψ besides those comprised in (28); in particular for the lath and parabola functions, the $\psi_{(m)}$ functions are given by $F_3(m)$ and $M_2(m)$ respectively.

In the language of physics, $\phi(x)$ represents a pulse and $\psi(m)$ represents the distribution of the square root of the energy in the (normal frequency) spectrum of the pulse. The $(\sin\pi x)/\pi x$ function throws all the energy into frequencies, n, less than p and distributes it uniformly over that region. The solutions comprised in (28) take some of this $(\text{energy})^{\frac{1}{2}}$ from frequencies $n<p$ and distribute it symmetrically over the region $p<n<2p$, but give none to any higher frequencies; while the lath, parabola ϕ_g, ϕ_h, and ϕ_k functions put some—though extremely little—of the energy into frequencies extending to infinity. However unnatural it may seem to use pulses containing harmonics of infinite frequency, it remains a fact that it is the only way of getting pulses without unnecessarily large oscillations at a large distance from the centre of the pulse, as will be seen by comparing any of these five functions with any pulse which can be obtained from (28) whatever function is taken for H.

Since ordinary harmonic analysis gives no higher harmonics than the pth, correction factors cannot be directly applied to any higher-order harmonics. It can be shown, however, that the higher-order coefficients are given by

$$a_{p+n}=a_{p-n}\,;\ a_{2p+n}=a_n\,;\ \text{and}\ b_{p+n}=-b_{p-n}\,;\ b_{2p+n}=b_n\,^*.$$

The ordinary harmonic analysis may be regarded as making use of these extended coefficients, but multiplying them by unity for $n<p$, by $\frac{1}{2}$ for $n=p$, and by zero for $n>p$.

* *Loc. cit.* p. 104.

The following table summarises the correction factors to be applied to harmonics below the $2p$th order according to the different functions which we have investigated. When 12 ordinates per half period are used for the harmonic analysis, the figures in the table may be used directly, but for any other number of ordinates a graph of the selected function must be drawn and ordinates erected at the appropriate values of n/p. In Table III. A are given the correction factors from (13 a), denoted by $E_2(n/p)$, from (13 b), denoted by $E_3(n/p)$, together with those for the lath and parabola functions extended from Tables I. and II. Table III. B gives similar results for the functions ϕ_g, ϕ_h, and ϕ_k for the previously-selected numerical values of g, h, and k. The last column, which gives practical agreement with the lath-function, $F_3(n/p)$, is obtained by taking $k^2=0{\cdot}0225$, which adopts as the ϕ_k function

$$e^{-x^2}\frac{\sin 20x/3}{20x/3}$$

on reducing the ordinate interval to $3\pi/20$. This last column may be recommended for adoption in practical work.

Table III. A.

$12n/p$.	$E_2(n/p)$.	$E_3(n/p)$.	$F_3(n/p)$.	$M_2(n/p)$.
0.........	1·000	1·000	1·000	1·000
1.........	1·000	1·000	1·000	1·000
2.........	·999	·999	1·000	1·000
3.........	·994	·995	·999	·998
4.........	·982	·987	·998	·995
5.........	·959	·969	·994	·987
6.........	·921	·939	·985	·973
7.........	·867	·897	·969	·947
8.........	·796	·836	·935	·905
9.........	·709	·765	·877	·841
10.........	·612	·682	·785	·754
11.........	·509	·589	·652	·643
12.........	·405	·493	·493	·516
13.........	·308	·397	·334	·390
14.........	·221	·308	·204	·275
15.........	·149	·227	·116	·182
16.........	·092	·158	·058	·113
17.........	·053	·104	·028	·066
18.........	·026	·063	·012	·036
19.........	·010	·034	·005	·018
20.........	·003	·016	·001	·008
21.........	·000	·006	·000	·003
22.........	000	·001	·000	·001
23.........	·000	·000	·000	·000
24.........	000	·000	·000	·000

Table III. B.

$12n/p$.	$Ag(n/p)$.	$Ah(n/p)$.	$Ak(n/p)$.	Suggest adopt.
0.........	1·000	1·000	1·000	1·000
1.........	1·000	1·000	1·000	1·000
2.........	1·000	·999	1·000	1·000
3.........	·999	·998	1·000	1·000
4.........	·998	·997	1·000	·999
5.........	·995	·994	·998	·997
6.........	·990	·987	·994	·991
7.........	·978	·976	·981	·975
8.........	·954	·954	·952	·942
9.........	·906	·911	·894	·881
10.........	·820	·832	·798	·784
11.........	·681	·695	·662	·653
12.........	·500	·500	·500	·500
13.........	·319	·305	·338	·347
14.........	·180	·168	·202	·216
15.........	·094	·089	·106	·119
16.........	·046	·046	·048	·058
17.........	·022	·024	·019	·025
18.........	·010	·013	·006	·009
19.........	·005	·006	·002	·003
20.........	·002	·003	·000	·001
21.........	·001	·002	·000	·000
22.........	·000	·001	·000	·000
23.........	·000	·000	·000	·000
24.........	·000	·000	·000	·000

Some New Interpolation Formula.

The lath-function, or any of the ϕ functions, may be used for interpolation purposes when we know the values of equally-spaced ordinates of a function and have at least six ordinates on each side of the point for which we require to calculate the function. Let $(0, y_0)$, $(1, y_1)$ . . . (n, y_n) be the tops of the given ordinates, and suppose x, the value for which we require to calculate the function, lies between the integers q and $q+1$; then it is easily seen that

$$\begin{aligned} y_x = & y_q\phi(q-x) + y_{q+1}\phi(q+1-x) \\ & + y_{q-1}\phi(q-1-x) + y_{q+2}\phi(q+2-x) \\ & + \cdot \cdot \cdot \cdot \cdot \end{aligned}$$

may be taken as the value of the function, provided that the terms become small before we have exhausted all the ordinates. The convergence and the accuracy will be increased if the first and last points be joined by a straight line to which the other points are then referred.

This interpolation formula requires only a table or graph of the selected ϕ function and a slide-rule to perform the multiplications.

Summary.

This paper deals with the subject of its title for the cases of both artificial and empirical functions. It is shown that in the case of artificial functions which only possess discontinuities of a single order, the exact harmonic analysis can be found without evaluating the integrals for the Fourier constants, by using the ordinary method of harmonic analysis by selected ordinates, and multiplying the results by certain definite functions.

In the case of empirical functions, the result is shown to depend upon what function we *choose* to represent the case in which all the selected ordinates are zero save one. The ordinary analysis works on the $(\sin x)/x$ function; other and better functions, both artificial and analytic, are suggested, including the function given by a flexible lath made to pass through the points and hence called the lath-function. All these different functions give a remarkable measure of agreement in the factors which they indicate the coefficients of ordinary harmonic analysis should be multiplied by.

X. *The Action of Heat on Pleochroic Halos.* *By* J. H. J. Poole, *Sc.D.**

[Plates I. & II.]

It has been known for many years that pleochroic halos in biotite disappear if the mica is heated for a short time to a dull red heat. When it was established that these halos are due to radioactive staining of the mica produced by the small amount of uranium present in the nucleus of the halo, it was naturally assumed that their disappearance, when heated, was analogous to similar effects observed in glass and various minerals. If glass is exposed for any length of time to either α-, β-, or γ-rays it becomes coloured, and it is found that on heating, in some cases, to quite moderate temperatures the colour is completely destroyed. During this process the glass becomes thermo-luminescent, but this property is only of a very short duration. Some observations of J. R. Clarke's (Phil. Mag. April 1923) may be quoted

* Communicated by Prof. J. Joly, F.R.S.

in this connexion. He found that glass became thermo-luminescent at 110° C., and that the luminescence lasted for about 13 minutes at this temperature. The glass was also completely decolorized at the end of this period. At higher temperatures the thermo-luminescence was more brilliant but of shorter duration, and the glass also lost its colour more rapidly. Some observations I have made myself completely agree with these results. We would accordingly expect that as the halo is also a radioactive staining of the mica, it would also be decolorized by heat, and probably become thermo-luminescent in the process. The micro-photographs shown in this paper prove, I think, fairly conclusively that the halo does not lose its colour when heated, and that its apparent disappearance is due to another cause. I may say also that I have totally failed to detect any thermo-luminescence in the halos. As, however, it apparently takes a dull red heat to have any effect on the visibility of the halo, it may be argued that the halo is thermo-luminescent at this temperature, but that the effect of the thermo-luminescence is swamped by the natural temperature radiation of the mica. It would be quite impossible to disprove this view, as the amount of energy liberated in thermo-luminescence is so excessively small; but, at any rate, at temperatures below this the halo is certainly not thermo-luminescent.

My attention was first directed to the action of heat on halos by the large number of reversed halo effects so often found in Ytterby mica. This mica is itself intensely dark, and in it, as well as the usual dark halos, there occur occasionally halos which are lighter in colour than the surrounding mica. More usually the reversed coloration of the mica occurs in streaks down the middle of which a line of small particles can often be seen. There seems to be little doubt that this bleaching of the mica is also due to the presence of uranium in the centre particles, but it is not very easy to explain why sometimes lightening of the mica should be produced by the same agent. One explanation that occurred to us was that possibly a halo if subjected to a high temperature after formation might become reversed. The Ytterby mica is of great geological age and comes from a highly-metamorphosed region, so that it was quite likely that the mica might have been raised to a high temperature at some period long subsequent to its original formation.

To test this theory a series of experiments on the effect of heating various specimens of mica to high temperatures in an electric furnace was tried. To measure the temperature

of the furnace a Calendar self-recording platinum thermometer was employed, and it was also arranged that this thermometer automatically controlled the temperature of the furnace, so that the mica could be kept at any desired temperature. The mica was usually wrapped in platinum foil and fastened on to the end of the platinum thermometer. Temperatures ranging from about 500° C. to 1000° C. were tried. It was found that an hour's heating at 550° C. did not affect the halos, but that after 15 minutes' heating at 610° C. they had completely disappeared. Higher temperatures always completely destroyed the halos, but in no case could any sign of the halos becoming reversed in tone be detected. It was observed that in all cases in which the halos disappeared the mica itself became very much shattered; in fact, when fairly large pieces of mica were employed, the expansion of the mica was sometimes sufficient to actually tear the platinum foil in which it was wrapped. This suggested that the mica actually becomes dehydrated at these temperatures, and that its break up is due to the explosive liberation of steam. Further experiments have confirmed this view, and have also thrown light on the probable mechanism of the formation of the halos.

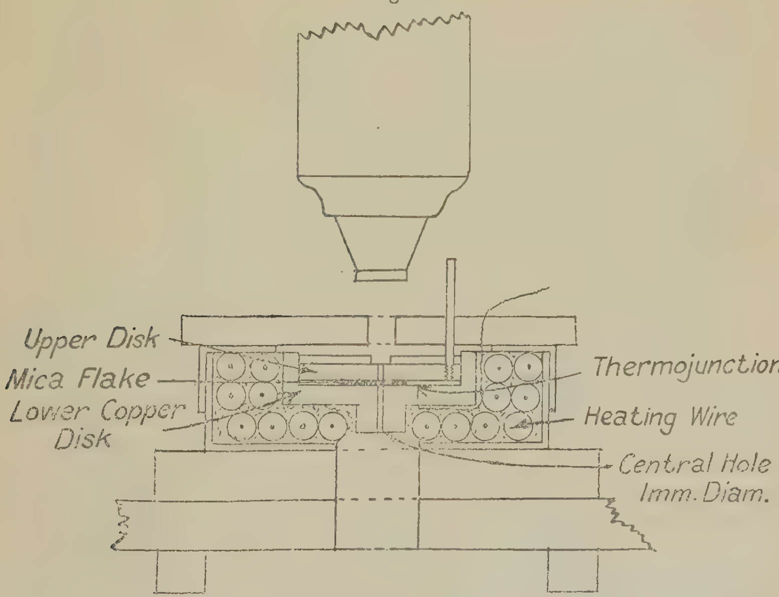
It was now felt that it would be highly desirable to be able to actually see what happened to the mica during the process of heating. A small vertical tube electric furnace was accordingly constructed. The lower end of the furnace tube was closed with a disk of fused silica, and the furnace was so small that it could be placed on the stage of an ordinary petrological microscopic, a long-focus object-glass (2 in.) being employed. The flake of mica under inspection was supported by a platinum framework half-way up the tube, and the halo viewed by light transmitted from below through the fused silica disk. As the latter was not in the same plane as the mica, flaws in it did not confuse the image in the microscope. No method of measuring temperatures was fitted, since the furnace was not very suitable for this purpose.

Not very many experiments were made with this arrangement, as it was not very convenient to use, but sufficient work was done to show that the idea that halos disappear by reverting like glass to the original colour of the mica was entirely incorrect. What actually occurs is that at about a dull red heat the whole surrounding mica begins to decompose and darken in tone, and if the heating is persisted in, the mica assumes the same colour as the halo, and hence, of course, the latter becomes invisible. Both Ballyellen and

Arendal micas behaved in this fashion. In no case could any sign of the halo itself becoming lighter in tone be detected.

These experiments were carried out in the spring of 1922, but no detailed account of them was published at that time. However, as the suggestion by Prof. Evans that possibly a large amount of the energy liberated in the earth's crust was used up in forming these halos lent renewed interest to the question, it was decided to construct a new form of furnace in which the temperature could be fairly accurately measured, and also microphotographs of the halo at various

Fig. 1.



temperatures taken. A sketch of the final form of furnace used is shown (fig. 1). The mica flake is held between the lower metal disk and the upper metal lid, and the halos are viewed by means of the small concentric holes in both disk and lid. As a source of light a pointolite lamp was employed. A platinum platinum-iridium thermocouple inserted into a small hole in the lower disk was used to measure the temperature. This thermocouple had been previously standardized by Johnson Matthey & Co., but it was thought advisable to check its readings by means of standard melting-points. The values obtained were practically identical with the

makers' values. The chief uncertainty in the temperature measurements arises from the fact that, owing to the necessity for having small holes in the upper and lower walls of the furnace to view the mica through, it is possible that the latter might be at a considerably lower temperature than that read by the couple. In most cases, however, the mica flake was large compared with the size of the aperture and completely covered it, so that no actual draught could pass through the furnace. When it was necessary to work with smaller mica flakes, or with halos near the edge of the flake, a thin sheet of muscovite or glass was used to support the mica. On this account the temperatures given must always be considered as the maximum temperatures which the mica could have attained. I do not think that the error introduced can be very large, as an examination of the furnace by its own light when heated under the microscope showed no sign of the central hole containing the mica flake, the whole field of the microscope being apparently equally bright. It may be noted that the temperatures taken to completely destroy the halo are slightly higher than those obtained with the first furnace, as we would expect.

The results obtained agree with those already quoted. Plate I. shows the effect of heat on two specimens of Ballyellen mica, the upper six microphotographs showing one piece of mica, and the lower three the second piece. The temperatures which the mica had attained when the exposure was made are given under the illustrations. In every case the same length of exposure was given so that the photos represent fairly accurately the gradual darkening of the mica. The time of heating for both specimens was about one hour; at a temperature of 700° C. they both became entirely opaque. The first sign of darkening of the Ballyellen mica usually sets in at about 420° C., but there is in no case much effect below 500° C. The effect is, of course, more apparent in the thicker flakes, and this leads to some pieces of mica darkening at lower temperatures than others. The final darkening of the mica is usually very rapid when once it commences. The photo taken of the second specimen at 640° C. is of some interest, as it shows how the radium C ring of one of the halos has actually been intensified by the heating. Many other photos and visual observations on the Ballyellen mica were made, but the two given may be taken as typical. The temperature at which the mica became finally practically opaque varied somewhat from specimen to specimen, the extremes being about 610° C. and

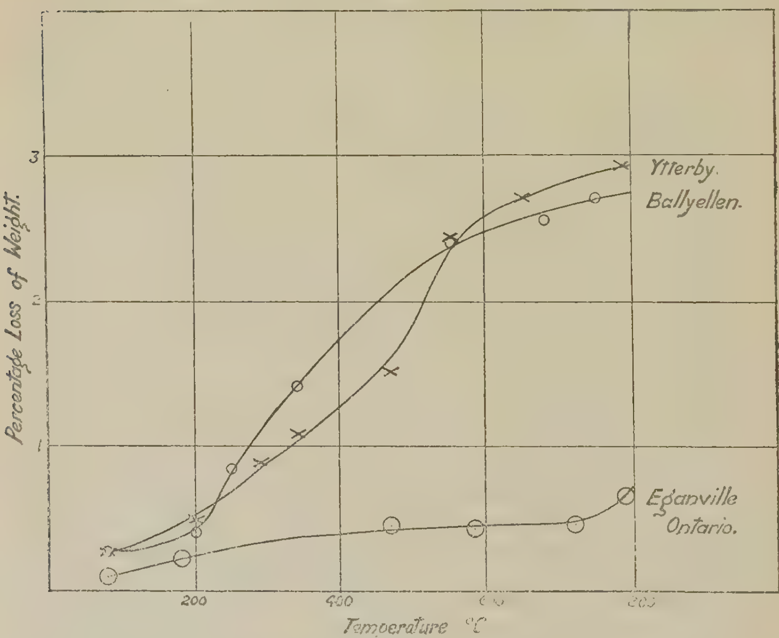
700° C. There is no reason to doubt, however, that prolonged heating at 600° C. would render the Ballyellen mica opaque.

The first four photos in Plate II. illustrate the effect on reversed halos in Ytterby mica. The chain of reversed halos shown are near the edge of a flake. This mica even at air temperature is very opaque, and, as it has a rather poor cleavage, it is much harder to deal with than the Ballyellen mica ; but I think the photos show quite clearly that reversed halos also disappear on heating, due to a progressive blackening of the mica. That the mica itself is actually darkening in colour can be seen from a comparison of the appearance of the small prominence on the mica flake at 15° C. and 505° C. It may be noted that the temperature required to destroy the reversed halo effects in Ytterby mica is considerably lower than that necessary for the Ballyellen positive halos. Other observations have confirmed this result. The last four photos in the plate show that prolonged heating at lower temperatures will blacken the mica slowly. The mica used had been originally heated to a temperature of 450° C. in the first furnace employed in 1922. The first photo at air temperature shows that this had not affected it much. The second was taken after 20 minutes' gradual heating when the temperature in the new furnace had reached 485° C. After a further half hour's heating no further change could be observed by visual observation. The third photo, taken an hour later at a slightly higher temperature, shows further darkening going on. The last exposure was made 10 minutes afterwards, when the temperature had been raised to 590° C., and demonstrates how rapidly the mica darkens with increased temperature. All traces of the halos finally disappeared at a temperature of 625° C. a few minutes later.

There is very little doubt, I think, that the darkening of the mica, when heated, is due to the loss of water. Micas always contain a certain percentage of water, amounting in some cases to as much as 4 per cent. by weight. The shattering of the mica observed in the earlier experiments had already led me to adopt this view, and later trials on the loss of weight of various types of mica when heated have confirmed it. Furthermore, that this loss is actually due to the evolution of water has been checked by heating the mica in a slow current of dry air and then absorbing the resulting water in a weighed calcium chloride tube. The following graph (fig. 2) shows the percentage loss in weight of various

micas plotted against the temperature to which they have been raised. It is of some interest, as in the Ytterby and Ballyellen micas, which lose water more easily than the Canadian, the halos also disappear at lower temperatures; in fact, in the latter mica, darkening only commenced at 690° C., and there was very little difference produced in its appearance by heating up to 790° C. In this respect it differs completely from the other two micas, which become entirely opaque and otherwise much altered when subjected to the same treatment. The chief change produced in the Canadian

Fig. 2.



mica is that it loses its elasticity, becoming very brittle; its opacity is only very slightly increased. The curves also show a general parallelism between the gradual loss of water with heating and the gradual darkening of the mica.

If this theory of the cause of the changes produced in the biotite by heating is correct, it naturally suggests that the original formation of the halo by α-rays from the nucleus is brought about in a very similar fashion. When α-rays enter either water or ice, they decompose it to a certain extent into hydrogen and oxygen. It seems highly

probable that they would also decompose the water present in the biotite, which is apparently only very weakly held, and thus indirectly produce dehydration and its resultant darkening of the mica. We can, moreover, calculate on certain assumptions how many α-rays per gram. would be required to completely dehydrate the mica, and compare this figure with the result obtained experimentally. Since the range of an α-ray of given velocity in any material is approximately inversely proportional to the density, it is justifiable to assume that, if there is x per cent. of water present in the mica, x per cent. of the ions produced by the α-rays will be produced in the water molecules. Accordingly, the number of α-rays required to completely dehydrate the mica is independent of the percentage by weight of water present, and is the same as the number required to decompose an equal mass of water. Now, the number of water molecules per gram. is about $3\cdot3\times10^{22}$, and the average number of ions produced by the α-rays due to the decay of radium emanation is 5×10^{5}. Unfortunately, the value we should adopt for the ratio between the number of pairs of ions produced and the number of water molecules decomposed is rather uncertain: thus Duane and Scheuer* found that in the liquid state the value was approximately unity, but in the solid it fell as low as $\frac{1}{20}$. We do not know which of these limits would be most likely to be approached by the water contained in the mica, but, at any rate, the minimum number of α-rays required to dehydrate 1 gram. of the mica is $3\cdot3\times10^{22}/2\cdot5\times10^{5}=1\cdot3\times10^{17}$, and the maximum $2\cdot6\times10^{18}$. These figures can be compared with some quantitative measurements on the darkening of Ballyellen mica by exposure to α-rays from radium emanation published by Professors Joly and Rutherford in the Philosophical Magazine for April 1913. They found that $3\cdot7\times10^{13}$ α-rays per sq. cm. produced an amount of darkening comparable with that of the less-developed halos—say about 10 per cent. that of the fully-darkened mica. The penetration of the α-rays into the mica was on the average about $\cdot016$ mm.; and hence the number of α-rays required to produce an equal amount of staining in one gram. of mica (density of mica $=2\cdot93$ gram. per c.c.) would be $3\cdot7\times10^{13}/2\cdot93\times1\cdot6\times10^{-3}=8\times10^{15}$, or about 10^{17} α-rays to produce fully-darkened mica. This figure is, of course, only a very rough estimate, as the exact percentage darkening produced by the radium emanation can only be very crudely judged;

* *Le Radium*, x. p. 33.

but it agrees sufficiently well with the minimum number of α-rays required to decompose the water in the mica to show that there is no impossibility in the theory of halo formation proposed. A further point in its favour is that no halos are found either in the quartz or felspar constituents of the granite, which contain no water, nor in the the muscovite, which, although it does contain water, does not darken when dehydrated by heating. It might possibly be contended that the darkening of the mica was really due to the oxidation of the ferrous iron, which is present in all biotites in large quantities, to the ferric state in the presence of atmospheric oxygen. It may be remarked that the α-ray, by its decomposition of the water in the mica, would also be capable of producing such an oxidation. Experiments however on the effect of heating biotite in a reducing atmosphere of either coal-gas or hydrogen have shown that the darkening of the mica proceeds just as readily in these cases, which proves that the action cannot be one of simple oxidation.

It is of some interest to calculate on these assumptions how much of the energy of the α-rays from the nucleus would be used in chemical work. Duane and Scheuer have shown (*loc. cit.*) that the amount of available energy of the α-rays in liquid water actually accounted for by the chemical work done in decomposing the water is about 6 per cent. In ice, of course, it will be only one-twentieth of this amount. Taking, however, the larger figure for the energy used, and remembering that at the outside only 4 per cent. of the mica is water, we find that the maximum amount of energy which could be used in chemical work will be about $\frac{1}{4}$ per cent. We thus see that even if all the radioactive elements in the rocks were contained in the biotites, a ridiculous assumption, still only $\frac{1}{4}$ per cent. of the radioactive energy could be used up in the formation of the halos. This small loss of heat energy would have no detectable effect on earth history. It might, however, be possible, though highly unlikely, that chemical work was done to the same extent in an average rock as in an equal mass of water ; but even then only 6 per cent. of the total radioactive energy would disappear as chemical energy, and we would expect that a large part of this chemical energy would soon revert again to the thermal form by recombination of the products of decomposition. As a matter of fact, Duane and Scheuer are inclined to attribute the small amount of decomposition observed in ice to this effect. There is perhaps, however, no object in labouring

this point, as Lawson * has already pointed out the impossibility on many grounds of believing that any appreciable quantity of energy could be used in producing chemical changes in the rocks.

Iveagh Geological Laboratory,
October 1927.

Note.—This theory of the mechanism of halo genesis does not lend itself to a direct explanation of the formation of reversed halos. The simplest explanation of their origin that can be offered is that they are due to some secondary effect produced in the mica molecule by the α-ray. If we assume that this secondary change is only brought about in a few collisions between the α-rays and the molecule, the fact that reversed halos are very rare except in very old micas receives a rational explanation. It would seem justifiable to imagine that it might be necessary for the α-ray either to collide with an atomic nucleus or to pass comparatively close to one to effect this change; and this would only occur very rarely. It may be remarked that reversed staining has never, as far as I am aware, been produced in the laboratory. It is hoped to conduct some further experiments on this point.

XI. *The Reversal of Helium Lines. By* Dr. Toshio Takamine, *F.Inst.P., and* Mr. Taro Suga †.

[Plate III.]

§ 1. *Introduction.*

THE phenomenon of the reversal of spectrum lines seen when a Geissler tube is viewed end-on is familiar to spectroscopists. Beside the numerous experiments of previous workers, the recent investigations by McCurdy ‡, Wood §, and Merton ‖ have revealed many interesting features, bringing at the same time a few problems for our closer study.

* 'Nature,' cxix. pp. 277 & 703 (1927).
† Communicated by Frank Twyman, F.Inst.P., F.R.S.
‡ Phil. Mag. ii. p. 529 (1926) and Proc. Nat. Acad. xii. Sci. p. 231 (1926).
§ Phil. Mag. ii. p. 876 (1926).
‖ Phil. Mag. ii. p. 975 (1926).

Thus, for instance, it is often stated that the reversal is seen when we take an end-on view of a narrow capillary tube which opens suddenly into a wide tube. Thereby it is believed that the light from the capillary is absorbed in the wider portion, which is filled with less luminous vapour.

Doubts are, however, expressed about this view—that it is necessary to have less luminous vapour in order to see the line reversed. In the case of the H_α line of hydrogen, Wood showed that even if we use a tube in which the capillary comes very close to the front end so that the effect of the less luminous layer is precluded, still the reversal of H_α line is clearly seen. Again, in the case of the D_3 line of helium (λ 5876 Å.), Merton mentions that the reversal may not be due to the less-luminous layer at all. He showed that for this line, not only the end-on view, but also the side-on view of the capillary showed the reversal, provided that the current density is large enough.

The aim of the present work was to examine these points in detail.

§ 2. *Experiments.*

The present experiment deals with the uncondensed discharge in helium at pressures below 10 mm. of mercury.

For analysing the spectrum we used a Hilger 33 plate echelon of 9·3 mm. thickness, the steps of echelon being placed horizontally. After passing through the echelon, the light was dispersed by two large prisms of flint glass of 10 cm. height and 14 cm. base. The objective lens had a focal length of 1 metre. In this way we could obtain an echelon photogram covering the region from λ 7065 to λ 3889 within a length of about 15 cm.

The discharge-tubes were made either of quartz or of pyrex glass with tungsten electrodes, the helium being purified by charcoal and liquid air.

We examined firstly the case of exciting both the emission and absorption tubes which were placed in a line (Part I.), and later the case of exciting the emission tube alone Part II.).

Part I.—Emission and Absorption Tubes both excited.

After repeating McCurdy's experiments using a tube as described by him*, we have modified the arrangement in a few points.

Firstly, in the case of McCurdy's experiment the capillary portion and the wider tube were directly connected so that

* See fig. 1 in McCurdy's paper (*loc. cit.*).

we had always the same pressure of helium filling the two parts. In the present experiment we entirely disconnected these two; viz., a discharge-tube having a capillary of 1 mm. bore and 30 cm. length was placed behind another discharge-tube with 10 mm. bore and 80 cm. length. Those two tubes were placed in a line, and the image of the front end of the narrower capillary was projected on to the slit of the echelon spectroscope with a lens of fairly long focal length. Lenses used had focal lengths ranging from 30 to 50 cm.

Secondly, we employed direct-current excitation for these two tubes, besides various kinds of transformers and induction coils. A 1-kilowatt direct-current dynamo of 10,000 volts was used to excite the capillary portion and another 2-kilowatt dynamo of 1200 volts for exciting the wide tube. In this way we could avoid the complexity arising from the difference in phase which is inevitable when we use two independent transformers for exciting the emission and absorption tube. This point was discussed by Ladenburg* in a recent paper.

The results of the experiments were in general confirmative of McCurdy's work. The parhelium lines were strongly absorbed in low-pressure helium, while at high pressure, orthohelium lines were more strongly absorbed. In the present experiment the pressure of helium in the capillary tube was 7 or 8 mm. of mercury in most cases, and that in the wide tube 1 or 2 mm.

One remarkable feature that came to our notice was the great difference in the appearance of the green line $\lambda\,5016\,(\nu=2S-3P)$† compared with the yellow line $\lambda\,5876$ $(\nu=2p-3d)$, when, firstly, the capillary tube alone and, secondly, both the capillary and wide tube were excited.

When the capillary tube alone was excited, the reversal of $\lambda\,5016$ occurred only under favourable conditions, but $\lambda\,5876$ readily appeared reversed. On the other hand, when the wide tube was also excited, $\lambda\,5016$ showed strong absorption so that the line appeared as a very clear doublet, while for $\lambda\,5876$ there was no perceptible difference whether the wide tube was excited or not.

Further, on examining the spectrogram, it was found that $\lambda\,3889\,(\nu=2s-3p)$ behaved exactly like $\lambda\,5016$, while $\lambda\,6678$ $(\nu=2p-3D)$ was similar to $\lambda\,5876$.

Summarising, we may state as follows:—

The lines belonging to the principal series which are related with the metastable states $2s$ and $2S$ were very

* Phil. Mag. iii. p. 512 (1927).

† Notation after Paschen and Götze, 'Seriengesetze der Linienspektren.' (J. Springer, 1922.)

distinctly absorbed in passing through the wide tube, while the lines belonging to the diffuse series showed no perceptible absorption. The latter, on the other hand, were readily seen reversed, even when the capillary tube alone was excited.

It is to be remarked that this is in conformity with the observations of previous investigators. The fact that metastable state is closely connected with the amount of absorption was shown already by the work of Meissner* and Dorgelo† in the case of neon, and by McCurdy‡ in the case of helium.

In fig. 1 (Pl. III.) is reproduced an echelon photogram showing the reversal of D_3 line (a) compared with the unreversed one from a side-on view (b). It will be seen that not only the strong principal line λ 5875·6 ($\nu=2p_1-3d$), but also its companion λ 5876·0 ($\nu=2p_2-3d$) shows reversal, with the difference that for the latter the reversal is much less clear.

According to Houston §, the main line $2p_1-3d$ is a close doublet having $\delta\nu=0{\cdot}120$ cm.$^{-1}$

As stated by McCurdy, the sharp series lines were hardly seen reversed. We saw only one exception in the reversal of λ 7066 ($\nu=2p-3s$).

The resemblance of the manner of reversal for the lines belonging to one and the same series is striking. As an example, we may mention the case of $\lambda\lambda$ 6678, 4922, 4388, and 4144, or that of 5876, 4472, and 4026.

Figs. 2 and 3 (Pl. III.) show the reversal of λ 6678 and λ 5016 respectively.

When we compare the lines belonging to one and the same series, we notice that the lower members show reversal more distinctly, and as we go up to higher members it becomes less and less clear.

In the course of our experiment, we noticed a very good illustration of the effect already described by McCurdy—namely, that of current density in the wide tube on the width of reversal of λ 5016.

The capillary tube being excited at a certain brightness, the excitation of the wide tube made the line λ 5016 appear double, as stated before. Now, by varying the current density of the wide tube periodically, we could see the pulsatory variation in the width of this doublet very clearly.

Wood states in his paper that the dark lines we see in the

* *Ann. d. Phys.* lxxvi. p. 124 (1925).
† *Zeits. f. Physik*, xxxiv. p .766 (1925).
‡ *Loc. cit.*
§ Proc. Nat. Acad. Sci. xii p. 91 (1927).

case of the reversal of H_α are not truly dark, but they appear black only by contrast to the very bright background. This was well illustrated in the case of λ 5016 with the arrangement here described—namely, when the wide tube is alone excited, the line λ 5016 appears as a fairly bright emission line, but as soon as the capillary is excited, we see at once that the bright line changes into a dark line.

As stated by Paschen * and McCurdy †, the effect of impurity was very detrimental to clear reversal; thus when the capillary tube was not quite clean, D_3 showed less distinct reversal, and when the helium in the wide tube became impure, the line 5016 was not at all absorbed in passing through it.

Part II.—Emission Tube alone excited.

Reversal of the principal series lines as described in Part I., namely when we pass the light of the emission tube through the absorption tube, is clearly due to absorption, and, in fact, previous work by Meissner ‡, Dorgelo §, and others has already emphasized the close bearing of the metastable states on this phenomenon.

Whether the case of the diffuse series lines, namely the reversal seen with the capillary tube having no dead space, should also be due to the simple process of absorption may perhaps be questioned. At least it may be stated that the physical process causing the reversal in this case does not manifest itself so clearly as in the former case.

In Part II. we shall confine our study to the case in which the emission tube alone is excited.

In comparing the degree of reversal, it is convenient to take as a measure the amount of separation between the double lines caused by reversal. It is to be noted, however, that this amount, which we shall denote by $\delta\lambda$, depends to a certain degree on the amount of exposure in taking the photograph, and, further, the clearness of the reversal is very sensitive to the difference in the manner of electrical excitation, so that no great accuracy can be expected in this measurement. The yellow line D_3 was found to be most suited for the measurement, the amount of $\delta\lambda$ varying from 30 m. Å. to 190 m. Å. in this experiment.

Now, on measuring $\delta\lambda$ for D_3 line with different tubes

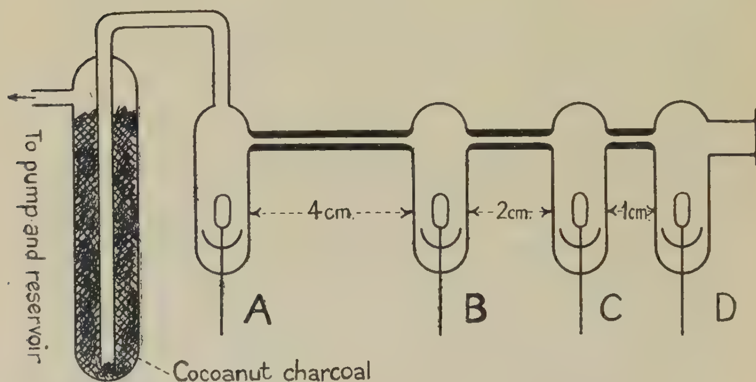
* *Ann. d. Phys.* xlv. p. 625 (1914).
† *Loc. cit.* ‡ *Loc. cit.* § *Loc. cit.*

under various conditions, we noticed that apparently the amount depended on the following factors :—

(i.) Length of the capillary.
(ii.) Pressure in the discharge-tube.
(iii.) Current density in the capillary.

As regards (i.), Buisson and Jausseran * verified this difference in an ingenious way by turning a tube obliquely to the line of sight. We prepared a tube as shown in fig. 4. One end of the transformer or generator was kept connected to A, while the other end was passed from B to C and D to change the length of the capillary. The result showed a steady increase in $\delta\lambda$ as the length of the capillary was increased.

Fig. 4.



As regards (ii.), we measured $\delta\lambda$ for the same discharge-tube at different pressures. With pressure less than 1 mm. the line was always sharp. It then began to show reversal at a few mm. pressure, and $\delta\lambda$ was usually largest at several mm. pressure. Above this point the line became too broad, making the reversal indistinct.

As regards (iii.), we prepared a tube such that the current flows through three capillary tubes, each 10 cm. long, having different diameters (1 mm., 2 mm., and 5 mm.) in succession. Those capillaries were placed parallel and close together so that the images of their ends could be projected on to the slit

* *C. R.* clxxx. p. 505 (1925).

of the spectrograph simultaneously. At a very small current density the line appeared sharp, and as we kept increasing it, the narrowest capillary first showed the reversal, and gradually the wider ones began to show the same effect. But here again, as in the case of (ii.), there was a practical upper limit of $\delta\lambda$, beyond which the line became too diffuse.

In short, we may state that the effect of the length of the capillary was the most important of the three factors.

During the examination of the above points, we noticed one interesting phenomenon which may be of importance in elucidating the process of reversal of this kind.

We took the end-on view of an ordinary Geissler tube of H form made by Götze in Germany which was filled with helium at about 10 mm. pressure. The capillary had a bore of 2 mm. and was 8 cm. long. The tube was excited by a Siemens induction coil of 15-cm. spark length, provided with an ordinary hammer interrupter. With a spark-gap of a few millimetres in series in the secondary circuit the discharge was found to be quite well rectified, and the line D_3 readily appeared reversed.

Now, the phenomenon which appeared worthy of note is that the reversal was seen decidedly more distinctly in the case when the electrode further away from the slit was made the cathode than when it was the anode. In the latter case there remained a certain amount of luminosity between the components of the doublet caused by reversal of D_3 which made the appearance rather indistinct.

Closer examination of the mode of discharge by means of a rotating mirror showed that there were a few wide striations near the cathode end of the capillary which became indistinct near the anode end.

The same difference due to polarity was observed when we excited the tube with direct current. The striations in the capillary were found also present in this case. We made two different tubes having the forms shown in figs. 5 and 6.

With the tube shown in fig. 5 the middle branch showed more distinct reversal when the back electrode A was anode compared with the other two branches, and *vice versâ*.

With the tube shown in fig. 6 all the three branches showed more distinct reversal with the cathode at A than at B. These differences are illustrated in figs. 7, 8, and 9 (Pl. III.), where the length-effect described in (i.) is also seen in that the amount of reversal differs according to the length of the capillary.

Fig. 5.

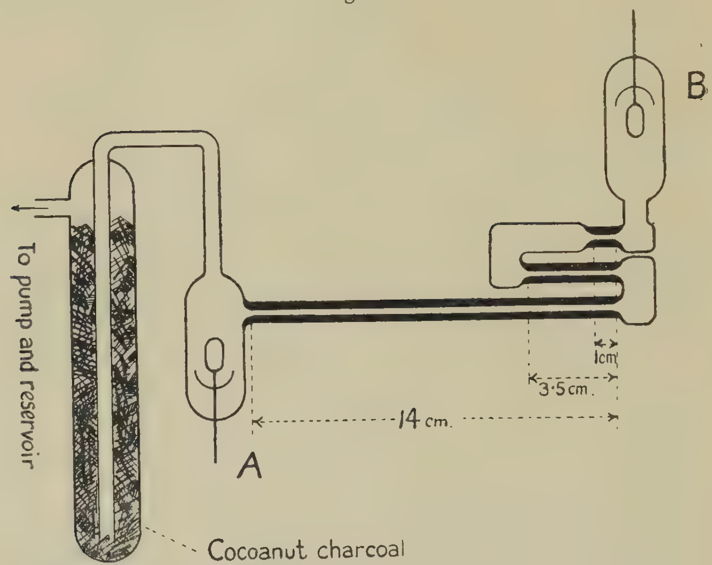
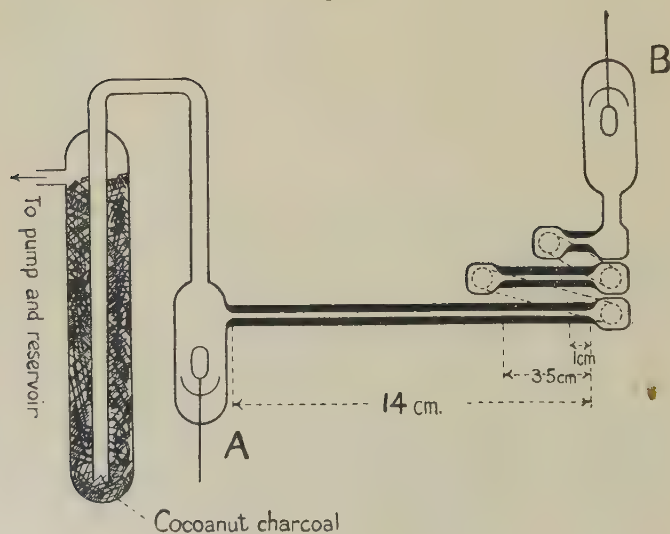

B
To pump and reservoir
1cm
3·5cm.
14 cm.
A
Cocoanut charcoal

Fig. 6.


B
To pump and reservoir
1cm
3·5cm
14 cm.
A
Cocoanut charcoal

In fig. 7 the image "*b*" due to the middle branch is more distinctly reversed, while in fig. 8 the opposite is the case. In fig. 9 all these branches show reversal. In all the figs, 7, 8, and 9 "*a*" corresponds to the shortest and "*c*" to the longest branch.

Altogether, this difference due to polarity seems to indicate the effect of heterogeneous distribution of the excited atoms along the length of the capillary. It may be remarked here that not only the heterogeneity along the tube, but that in the radial direction in the section of a tube, mainly due to the wall of the capillary, may also contribute to the phenomenon of reversal.

§ 3. *Summary.*

1. Reversal of helium lines was examined by an echelon grating, firstly when a long capillary tube was viewed end-on, and secondly when another long tube of wider bore was placed between the capillary tube and the spectrograph.

2. The lines belonging to the principal series and the diffuse series show reversal, but there is much difference in the manner of reversal for these two series.

The lines belonging to the principal series which are connected with the metastable states ($2s$ and $2S$) are distinctly absorbed in passing through the wider tube. In other words, the effect of a less-luminous layer is clearly seen for such lines as λ 5016 ($\nu = 2S - 3P$) and λ 3889 ($\nu = 2s - 3p$).

The lines belonging to the diffuse series, such as λ 6678 ($\nu = 2P - 3P$) and λ 5876 ($\nu = 2p - 3d$) are seen reversed with the capillary tube alone excited, while the effect of the wider tube is hardly noticeable in this case.

3. A distinct difference in the clearness of the reversal is noticed as regards polarity, or, in other words, according as to whether we pass the current from the front or the back end of the capillary.

The Institute of Physical and
Chemical Research, Tokyo,
July 1927.

XII. *The Spectrum of Ionized Sodium.* *By* F. H. Newman, *D.Sc., A.R.C.S., Professor of Physics, University College of the South West of England, Exeter* *.

1. *Introduction.*

THE spark spectra of potassium, rubidium, and cæsium in the visible and near ultra-violet regions have been investigated by many experimenters, but little work has been done on the first spark spectra of sodium, although Schillinger † and Foote, Meggers and Mohler ‡ have measured the wave-lengths of the more important lines. Among those listed by Schillinger are several belonging to the arc spectrum of the element. Foote and his collaborators used a three-electrode discharge-tube, and studied the different sets of lines emitted as the accelerating potential gradient was increased. The first ionization potential of sodium vapour is 5·1 volts approximately, and this is sufficient to excite the whole arc spectrum. At higher voltages the spark lines begin to appear. These experimenters noted that the spark lines appeared above 30 volts, and even at 5000 volts very few spark lines were ever observed which were not excited at 50 volts, although their intensity was increased considerably. More recently Mohler § found that with currents of 0·02 amp. the spark lines were entirely absent at 40 volts, three were barely visible at 42 and 43 volts, while many lines appeared at 45 volts. With 35 volts and 0·35 amp. the lines were bright, so he concluded that the spark lines were excited near 44 volts with small currents and about 10 volts lower with large currents.

From the theory of atomic structure, what might be termed the "spectral centre of gravity" of the sodium spark spectrum should be at a shorter wave-length than that of potassium. The latter is very rich in lines, and their wave-lengths have been carefully measured and classified by different workers ‖. Shaver ¶ extended the study of the alkali spectra in the extreme ultra-violet region by means of

* Communicated by the Author.
† *Akad. Wiss. Wien Sitz.-Ber.* cxviii. 11 (*a*) (1909).
‡ Astrophys. Journ. lv. p. 145 (1922).
§ Bureau of Standards Scientific Papers, No. 505 (1925).
‖ See de Bruin, *Proc. Konin. Akad. Van Weten te Amsterdam*, vol. xxix. No. 5 (1926).
¶ Roy. Soc. Canada, Trans. xviii. Sect. 3, p. 23 (1924).

the electrodeless discharge, using a fluorite vacuum spectrograph, in order to make additions to the lists of wave-lengths available for series analysis, but no sodium lines were found in the fluorite region. Millikan and Bowen *, using the explosive spark method, found only one definite line in the whole region studied (300 A. to 2000 A.), namely that at λ 372·3 A. They also found a line at λ 376·6 A. when pure sodium electrodes were employed, but this line was absent with sodium fluoride electrodes. The line λ 372·3 A. they attributed to the L spectrum of sodium, as a line is definitely predicted by theory at this wave-length and represents the orbital change of an electron from the M to the L shell when an electron is removed from the latter. Comparing the values of $\sqrt{\frac{\nu_L}{R}}$ for the L_{11} and L_{111} levels of aluminium, magnesium, sodium, and neon, λ 372·3 A. corresponds to the energy difference M_1-L_1; and so the value of the L_1 energy level in the sodium atom is fixed at $\nu=310{,}090$ or 38·3 volts. This agrees fairly well with the critical potential 38·9 volts found by Foote and Mohler †, and represents the work required to eject one electron from the rare gas shell of the ion, the M electron having been removed previously. The L_{111} limit computed from X-ray frequencies of the K series is 31·5 volts.

Accurate measurements of the wave-lengths of lines in the Na II spectrum are necessary and important, because, in accordance with the displacement law, this spectrum should resemble closely the arc spectrum of neon. The latter has been analysed by Paschen ‡ into four series of S terms, ten *p* sequences and twelve *d* sequences. The possible number of combinations is very large, but the selection principle prohibits several. In addition to these series, which followed the Ritz formula, he found other series with the limits displaced when calculated by means of the Ritz formula, as indicated by the combinations which gave the first lines of the series. The explanation of this spectrum requires the extraction of an electron from more than one sub-group of the L group, and the simultaneous movement of the two electrons. The series which are represented by the Ritz formula directly are triplets and quintets, while those series which require the addition of a

* Phys. Rev. vol. xxiii. p. 1 (1924).

† Bureau of Standards Scientific Papers, No. 425 (1921).

‡ *Ann. d. Phys.* lx. p. 405 (1919), and lxiii. p. 201 (1920).

constant before they can be represented by such a formula are singlets and triplets.

The sodium atom contains in its L group two L_{11} electrons, two L_{21} electrons, and four L_{22} electrons, the suffixes denoting the azimuthal and inner quantum numbers respectively. The single M electron being removed, one of the L_{21} electrons may be displaced to various higher levels and give rise to a singlet and triplet system. If, on the other hand, one of the L_{22} electrons in the closed sub-group is removed to different levels, a system of triplets and quintets should be produced. The L_{22} electron being less firmly bound than the L_{21} electron, the removal of the latter should leave the residue of the atom with a greater energy than is the case when the L_{22} electron is removed, so that the sequences arising from the removal of the L_{21} electron are those requiring the addition of a constant so that they may be expressed by a Ritz formula. In other words, one would expect the first spark spectrum of sodium to resemble the arc spectrum of neon as regards the multiples present, but with the series constant taking the value 4N.

Before any complete analysis of the Na II spectrum is attempted, accurate measurements of the wave-lengths are necessary.

2. *Experimental.*

The electrodeless discharge is the most convenient method of exciting the spark spectrum of sodium. A quartz tube, about 30 cm. long and 1 cm. in diameter, with a transparent quartz window sealed at one end was used as the discharge-tube. The metallic sodium, placed in a small side tube, was distilled into the main tube under a vacuum, the latter obtained by means of an ordinary Gaede mercury pump. After enough metal had been distilled, the side tube was sealed off, and afterwards the discharge-tube was sealed off from the pumps. By this means, and after preliminary baking of the main tube, all residual gases were eliminated. The tube, with about fifty turns of fine copper wire wound round it, was enclosed in a glass tube open at both ends, and then placed inside an electric heater. The glass enclosing the tube assists in maintaining a good insulation between the high-tension wire and the wire of the heater. In obtaining the electrodeless discharge, a 16-inch induction coil was used with its secondary joined in parallel with two large parallel plate condensers. The coil was operated by means of a Wehnelt interrupter joined to the 110 A.C. mains. The Wehnelt interrupter is very effective in the production of

this type of discharge. The lowest temperature possible, consistent with fair luminosity, was employed to keep the vapour-pressure low; otherwise the arc lines predominated. The radiation was yellow in colour, the D lines always appearing bright, but at low vapour-pressures the spark lines were intense. The radiation was focussed by means of a quartz lens on to the slit of a quartz spectrograph. Various types of plates were tested for sensitiveness, including Schumann plates, but no lines of greater refrangibility were obtained with these than on the ordinary type of plate. Although Schillinger recorded a line at λ 2138·4, and Foote, Meggers, and Mohler at λ 2318·0, no line below λ 2386·41 was photographed in the present work. Potassium spark lines below these limits have been measured, and in the spark spectrum of this metal many more lines are present than are recorded with sodium. In addition, many of the lines listed in Table I. are faint, and prolonged exposure did not materially increase their intensity. They were measured against a comparison iron-arc spectrum.

Table I.—Wave-lengths in the Na II Spectrum.

Intensity.	λ I.A. (Author).	λ (Foote, Meggers, Mohler).	λ (Schillinger).	ν (vacuum).
—	—	—	2138·4	—
—	—	2318·0	—	43127·2
1	2386·41	—	—	41891·1
1	2394·14	—	—	41755·9
1	2459·62	—	—	40644·5
0	2469·10	—	—	40488·5
0	2474·92	—	—	40393·1
6	2493·38	2493·36	2493·45	40094·1
1	2497·31	2497·3	—	40031·1
1	2506·27	2506·40	—	39888·1
0	2510·77	—	—	39816·4
1	2515·57	2515·50	—	39740·5
4	2531·64	2531·54	—	39488·2
0	2563·47	—	—	38998·1
0	2578·24	—	—	38774·6
1	2586·37	2586·37	—	38652·7
7	2612·10	2612·18	2612·27	38271·9
6	2661·65	2661·72	2661·82	37559·5
6	2671·90	2671·94	2672·04	37415·5
5	2678·25	2678·2	—	37326·8
6	2809·73	2809·76	2809·87	35580·2
1	2818·58	2818·5	—	35468·4

TABLE I. (*continued*).

Intensity.	λ I.A. (Author).	λ (Foote, Meggers, Mohler).	λ (Schillinger).	ν (vacuum).
1	2830·01	2830·0	—	35325·2
2	2839·40	2839·36	2839·47	35207·7
7	2841·99	2841·99	2842·10	35176·3
—	—	—	2856·27	—
3	2859·64	2859·56	2859·07	34959·2
0	2861·34	2861·2	—	34938·4
3	2871·04	2870·89	2871·00	34820·9
3	2871·53	2871·39	2871·50	34814·5
0	2873·13	2873·0	—	34795·1
5	2881·38	2881·33	—	34695·5
3	2886·42	2886·46	2886·57	34634·9
4	2894·19	2894·15	—	34541·9
3	2901·36	2901·39	2901·50	34456·5
6	2905·11	2905·16	2905·27	34412·0
6	2917·78	2917·76	2917·87	34262·6
5	2919·34	2919·46	2919·57	34244·3
4	2921·14	2921·14	2921·25	34222·9
2	2923·67	2923·65	2923·76	34193·6
—	—	—	2926·67	—
0	2931·62	2931·44	2931·55	34101·0
1	2934·38	2934·42	2934·53	34068·8
3	2938·06	2938·11	2938·22	34026·1
1	2945·92	2945·83	2945·94	33935·2
3	2947·56	2947·51	2947·62	33916·5
7	2951·50	2951·53	2951·64	33871·2
1	2952·83	—	—	33855·9
1	2960·11	2960·10	2960·22	33772·7
C	2971·17	2971·1	—	33647·0
5	2975·24	2975·20	2975·32	33600·9
0	2977·31	2977·52	2977·64	33577·6
5	2979·96	2979·92	2980·04	33547·7
0	2981·03	—	—	33535·1
5	2984·45	2984·44	2984·56	33497·2
2	3007·65	3007·71	—	33238·9
2	3009·60	—	—	33217·4
4	3015·80	3015·81	3015·93	33149·1
4	3029·56	3029·70	3029·82	32999·5
4	3037·32	3037·29	3037·41	32914·2
3	3045·97	—	—	32820·8
0	3050·43	3050·48	3050·60	32772·8
3	3053·93	3053·97	3054·09	32735·3
5	3056·30	3056·34	3056·46	32709·9
0	3061·72	3061·75	3061·87	32651·9

TABLE I. (*continued*).

Intensity.	λ I.A. (Author).	λ (Foote, Meggers, Mohler).	λ (Schillinger).	ν (vacuum).
1	3064·86	3064·8	—	32618·4
1	3066·73	3066·74	—	32598·6
4	3074·56	3074·63	3074·75	32515·6
5	3078·55	3078·51	3078·63	32473·4
1	3080·30	3080·58	3080·70	32455·0
8	3092·98	3092·91	3093·03	32321·8
3	3104·52	3104·6	—	32201·8
0	3107·69	3107·6	—	32168·9
3	3124·64	3124·63	3124·76	31994·5
7	3129·55	3129·57	3129·70	31944·3
4	3135·69	3135·65	3135·78	31881·8
0	3137·80	3138·11	3138·24	31860·8
0	3146·05	3146·07	3146·20	31766·7
4	3149·51	3149·48	3149·61	31741·3
5	3164·16	3164·11	3164·24	31594·9
1	3175·41	3175·38	3175·51	31482·9
2	3179·29	3179·19	3179·32	31444·5
5	3190·05	3189·94	3190·07	31338·4
5	3212·44	3212·42	3212·55	31120·1
0	3216·60	3216·5	—	31079·8
2	3226·16	3226·1	—	30987·7
3	3235·07	3235·1	—	30902·4
1	3251·22	3251·3	—	30748·9
5	3258·38	3258·36	3258·50	30681·4
0	3260·45	3260·4	—	30661·8
3	3274·36	3274·27	3274·41	30531·6
7	3285·86	3285·76	3285·90	30424·7
0	3305·28	3305·2	—	30245·9
3	3318·14	3318·2	—	30128·7
0	3320·70	3320·7	—	30105·5
3	3327·81	3327·9	—	30041·3
0	3345·47	—	—	29882·6
1	3363·29	—	3360·50 ?	29724·3
0	3373·26	—	3371·29 ?	29636·4
0	3381·22	—	—	29566·6
0	3385·61	—	—	29528·3
—	—	3400·2	—	29401·4
1	3405·00	—	—	29360·2
1	3462·68	3462·58	3462·73	28871·1
10	3533·03	3533·08	3533·23	28296·2
0	3608·90	—	—	27701·4
0	3619·20	—	—	27622·6
	3631·37	3631·31	3631·46	27530·0

TABLE I. (*continued*).

Intensity.	λ I.A. (Author).	λ (Foote, Meggers, Mohler).	λ (Schillinger).	ν (vacuum).
0	3634·30	3634·3	—	27508·0
0	3651·5	—	—	27379
0	3662·4	—	—	27297
0	3676·1	—	—	27195
3	3711·2	3711·15	3711·30	26938
0	3730·1	—	—	26801
0	3745·6	3745·6	—	26690
0	3798·6	—	—	26318
0	3802·2	—	—	26293
2	3816·8	—	—	26192
0	3849·0	—	—	25973
2	3870·7	—	—	25828
0	3881·1	3881·0	—	25759
0	3889·3	—	—	25704
0	3914·2	—	—	25541
0	3921·0	—	—	25497
0	3928·2	—	—	25450
0	3934·9	—	—	25407
0	3943·8	—	—	25349
0	3983·8	—	—	25095
0	4012·6	—	—	24915
0	4061·8	—	—	24613
0	4094·0	—	—	24419
0	4123·9	4124·0	—	24242
0	4157·3	—	—	24047
0	4178·3	—	—	23926
1	4347·5	—	—	22995
0	4447·0	—	—	22481
0	4455·0	—	4500·0	22441
0	4465·5	—	—	22387
0	4480·3	—	—	22314
—	—	—	—	—
—	—	—	4538·9	—
—	—	—	4542·7	—
—	—	—	4546·5	—
—	—	—	4551·5	—
—	—	—	4564·4	—
—	—	—	4569·4	—
—	—	—	4572·7	—
1	4581·0	—	4581·7	21823
0	4807·0	—	—	20797
2	4830·9 (K II)	—	—	20695

Table II.—Constant differences in the Na II Spectrum.

Frequency (ν).	Difference ($\Delta\nu$).
37559·5 37415·5	144·0
35468·4 35325·2	143·2
34959·2 34814·5	144·7
33916·5 33772·7	143·8
32598·6 32455·0	143·6
31482·9 31338·4	144·5
34244·3 34026·1	218·2
34412·0 34193·6	218·4
32735·3 32515·6	219·7
31994·5 31776·7	217·8
31338·4 31120·1	218·3
31120·1 30902·4	217·7
30748·9 30531·6	217·3
35580·2 35325·2	255 0
34193·6 33935·3	258·3
33855·9 33600·9	255·0
33497·2 33238·9	258·3
32772·8 32515·6	257·2
32709·9 32455·0	254·9
32201·8 31944·3	257·5
31594·9 31338·4	256·5
30681·4 30424·7	256·7

Frequency (ν).	Difference ($\Delta\nu$).
34959·2 34634·9	324·3
33871·2 33547·7	323·5
33238·9 32914·2	324·7
30987·7 30661·8	325·9
30748·9 30424·7	324·2
34938·4 34101·0	837·4
33547·7 32709·9	837·8
32321·8 31482·9	838·9
31741·8 30902·4	839·4
27530·0 26690·2	839·8
39888·1 37559·5	2328·6
39740·5 37415·5	2325·0
35325·2 32999·5	2325·7
34068·8 31741·8	2327·0
33772·7 31444·5	2328·2
35176·3 32709·9	2466·4
34634·9 32168·9	2466·0
34412·0 31944·3	2467·7
32999·5 30531·6	2467 9
31338·4 28871·1	2467·3

TABLE II. (*continued*).

Frequency (ν).	Difference ($\Delta\nu$).	Frequency (ν).	Difference ($\Delta\nu$).
38652·8 35580·2	3072·5	39740·5 32455·0	7285·5
34814·5 31741·8	3072·7	39488·2 32201·8	7286·4
34193·6 31120·1	3073·5	38271·9 30987·7	7284·2
31944·3 28871·1	3073·2	37326·8 30041·3	7285·5
34412·0 31338·4	3073·6	35580·2 28296·2	7284·0
33497·2 30424·7	3072·5	34814·5 27530·0	7284·5
32473·4 29401·4	3072·0	34222·9 26938·2	7284·7

TABLE III.—Frequency differences in the Na II Spectrum.

No.	Intensity.	λ (I.A.) in air.	Frequency in vacuum (ν).	Difference ($\Delta\nu$).
1	—	—	—	—
2	—	—	—	—
3	—	—	—	—
4	7	2841·99	35176·3	217·1
5	3	2859·64	34959·2	696·6
6	6	2917·78	34262·6	327·3
7	1	2945·92	33935·3	1316·9
8	1	3064·86	32618·4	736·6
9	4	3135·69	31881·8	
1	1	2497·31	40031·1	2471·6
2	6	2661·65	37559·5	2091·1
3	1	2818·58	35468·4	
4	—	—	—	530·0
5	0	2861·34	34938·4	694·1
6	5	2819·34	34244·3	327·8
7	3	2947·56	33916·5	1317·9
8	1	3066·73	32598·6	737·8
9	0	3137·80	31860·8	
1	1	2506·27	39888·1	2472·6
2	6	2671·90	37415·5	2090·3
3	1	2830·01	35325·2	
4	—	—	—	530·1
5	0	2873·13	34795·1	694·1
7	1	2931·62	34101·0	
7	—	—	—	1646·0
8	1	3080·30	32455·0	
9	—	—	—	—
4	6	2905·11	34412·0	218·4
5	2	2923·67	34193·6	696·4
6	5	2984·45	33497·2	

TABLE III. (*continued*).

No.	Intensity.	λ (I.A.) in air.	Frequency in vacuum (ν).	Difference (Δν).
4	3	3124·64	31994·5	
5	0	3146·05	31776·7	217·8
6	0	3216·60	31079·8	696·9
7	1	3251·22	30748·9	330·9
4	5	3190·05	31338·4	
5	5	3212·44	31120·1	218·3
6	7	3285·86	30424·7	695·4
6	2	3007·65	33238·9	
7	4	3037·32	32914·2	324·7
8	5	3164·16	31594·9	1319·3
5	5	3078·55	32473·4	
8	3	3318·14	30128·7	2344·7
9	0	3400·2	29401·4	727·3
4	2	3009·60	33217·4	
5	4	3029·56	32999·5	217·9
8	0	3260·45	30661·8	2337·7

A preliminary analysis of the results shows no lines or sequences obeying the Ritz formula, but there are many pairs of lines in the spectrum having constant differences. These have been collected and tabulated in Table II. In addition many of the strong lines can be arranged in groups of nine lines having constant wave-number separation, although all of these groups are incomplete. They are given in Table III. Owing to the incompleteness of the groups and the fact that many of the observed lines are not represented by any of the terms of these groups, no term scheme has been arranged.

Hertz * found two strong resonance lines in the neon spectrum at λ 735·7 ± 0·5 and λ 743·5 ± 0·5. Their frequency difference is 1428 ± 3, and agrees with the difference between the Paschen terms 1_{s_2} and 1_{s_4}, so that they are to be regarded as combinations of the fundamental term with two new terms. The fundamental term must be the optically undiscovered one at approximately 174,000 cm^{-1}. It has been noted previously that Millikan and Bowen found the two lines λ 372·3 and λ 376·6 in the spark spectrum of sodium. Their frequency difference is 3075. If these two lines are analogous to the two neon resonance lines, then 3075 should be an important frequency difference among the Na II lines. Seven pairs of lines with a difference 3073 ± 1 have been discovered, and are shown in Table II. Further investigation in the extreme ultra-violet of the Na II spectrum seems very necessary.

* *Zeits. f. Physik*, xxxii. 11–12, p. 933 (1925).

XIII. *Notes on the Resonances of a Violin.* *By* Florence M. Chambers, *M.Sc., Junior Lecturer in Physics, Queen's University, Belfast* *.

Introduction.

THE following investigation was undertaken with the object of finding resonance pitches of a violin, in the hope of throwing light on the "Wolf-note" phenomenon.

As described in an earlier paper †, when a steel wire is mounted on a light metal bar and made to vibrate, two kinds of resonance are found : one associated with the coincidence of an odd mode of the string (or bar) with an even mode of the bar (or string) respectively, and the other with the coincidence of an even (or odd) mode of the string with an even (or odd) mode of the bar respectively.

It is rather a jump from the case of a uniform bar and string to that of a violin, but it was thought that in the case of the violin something corresponding to one or other of the string-bar resonances might lie at the root of the "wolf-note" phenomenon.

As will be seen, resonance pitches were found for the violin ; but the behaviour of string and violin bears little resemblance to that of the string and bar.

The method used was to measure at different pitches the resonance amplitude of a steel wire stretched on the violin and maintained in vibration by an applied alternating force of constant amplitude, as described in an earlier paper ‡. On a rigid support the amplitude of vibration of the wire would change more or less gradually with changing pitch (as the damping is usually greater at higher pitches). On a violin the amplitude rose and fell in an apparently arbitrary manner, and graphs of amplitude and pitch showed a large number of definite minima. These minima occurred at the same pitches for different modes

* Communicated by Prof. W. B. Morton, M.A.

† W. B. Morton and F. M. Chambers, "On the Combined Vibration of a Bar and String, and the Wolf-Note of a Stringed Instrument," Phil. Mag. vol. l. p. 570 (1925).

‡ F. M. Chambers, "Application of a Thermionic Valve to the Measurement of the Damping of Vibrations of a Steel Wire," Phil. Mag. xlviii. p. 636 (1924).

of getting the string to give the note, indicating that they are functions of the violin itself.

When the note given by the wire approaches a natural frequency of part of the violin, (1) the part is set into vigorous vibration and the energy to maintain this vibration comes from the string ; (2) there is an increase of frictional damping in the motion of the wood ; so that, on the whole, there should be a fall in amplitude of vibration of the wire.

Hence the minima seem to indicate the pitches of natural frequencies of the violin.

Experiments.

The gut strings were removed, and a fine steel wire stretched in the place usually occupied by the A string. The oscillating current from a valve circuit, which could be tuned to any desired frequency, was passed through the wire, and this was maintained in vibration by the reaction between the current and the field of a strong electromagnet between whose poles it was placed. Care was taken that the amplitude of the oscillating current, and therefore the alternating force applied to the wire, remained constant throughout a set of observations.

An auxiliary wire, stretched by a weight on a wooden sonometer and through which the oscillating current also passed, was used as a standard of pitch. This wire was placed between the poles of another electromagnet. It was bridged to a suitable length, and then the frequency of the current adjusted so that there was maximum response. The electromagnet of the standard was turned off, that belonging to the violin wire was turned on, and the latter screwed up until it resonated. To ensure that the pitch of the standard was exactly the same as the pitch of the violin wire, slight further adjustments of current and length of standard were usually necessary. The amplitude of vibration of the violin wire was read off on the eyepiece scale of a microscope.

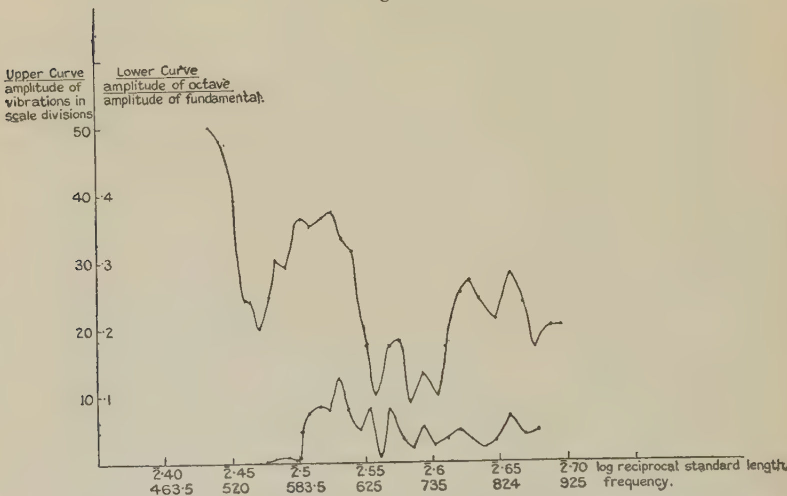
The pitch is inversely proportional to the standard length. In order that a given length of abscissa should represent a given interval, the amplitude of vibration was plotted against the log. reciprocal of the corresponding standard length.

To get higher pitches than could be obtained by tension of the open string alone, a small wooden bridge was put under the wire on the fingerboard, and the tuning for subsequent observations carried out by tension as before.

Results.

The results found are outlined above. The upper curve of fig. 1 shows part of a typical amplitude-pitch graph, the observations for which were made with the string giving its fundamental note. A large number of these graphs were plotted for each of the two violins used, each graph corresponding to a definite mode of the string. It was found that in whatever mode the string was vibrating, and whether it was bridged or not, the minima for the same violin always occurred at about the same pitches.

Fig. 1.



The fact that the minima are not always exactly in the same place may be accounted for partly by the nature of the standard used—a sonometer wire. It probably also depends on changes of the atmosphere etc., which would have the effect of making slight alterations in the elasticity of the different parts of the violin.

Further Experiments.

In order to approach the matter from another angle, a series of 27 vibration photographs of the same point on the wire were taken, after the manner of Krigar-Menzel and Raps, for different pitches ranging over $\frac{2}{3}$ of an octave. The string was vibrating in its fundamental mode. Each photo was enlarged, and the enlarged curves analysed by a Mader harmonic analyser.

It was found :

(1) The preponderating tone in all the curves is the fundamental.

(2) The highest amplitude of the first overtone is about 12 per cent. of that of the fundamental. The lowest is zero.

(3) The highest amplitude of the second overtone is 4 per cent. of the corresponding fundamental.

(4) The highest amplitude of any other overtone is less than 2 per cent. of the fundamental.

The results of the harmonic analysis are not reliable within 1 per cent.

An additional test of the conclusion that the pitches of the minima are functions of the violin itself is provided by examining the amount of first overtone in the vibration curve of the string. Assuming that the first harmonic is present in a constant proportion in the alternating applied force, we should expect its presence in the forced vibration of the string to be determined in the same way as for the fundamental ; the minima of the amplitude of the octave in the complex note should occur at the same pitches. This was found to be the case for the range examined, as is shown in fig. 1. In order to reduce all the amplitudes of the first overtone to the same scale, each was expressed as a fraction of the corresponding fundamental amplitude (*i. e.* approximately, as a fraction of the whole amplitude).

It now remains to trace to which parts of the violin the resonances belong. It is easiest to tackle first the enclosed air.

Helmholtz * found, on a violin which he examined, resonances at two pitches between c' and $c'\sharp$; the other between a' and $a'\sharp$. He got them by listening at the back of a violin while a piano was being played, and by noticing that some notes were more strongly reinforced than others.

In these experiments a vibrating telephone diaphragm was used as a faint source of sound. The violin was disconnected, and the telephone connected across the oscillating circuit.

The receiver was held with its diaphragm opposite one of the F holes of the violin, and as the pitch of the oscillating circuit was varied, it was noticed that at certain pitches, corresponding to natural frequencies of the enclosed air, the faint sound from the diaphragm was considerably reinforced.

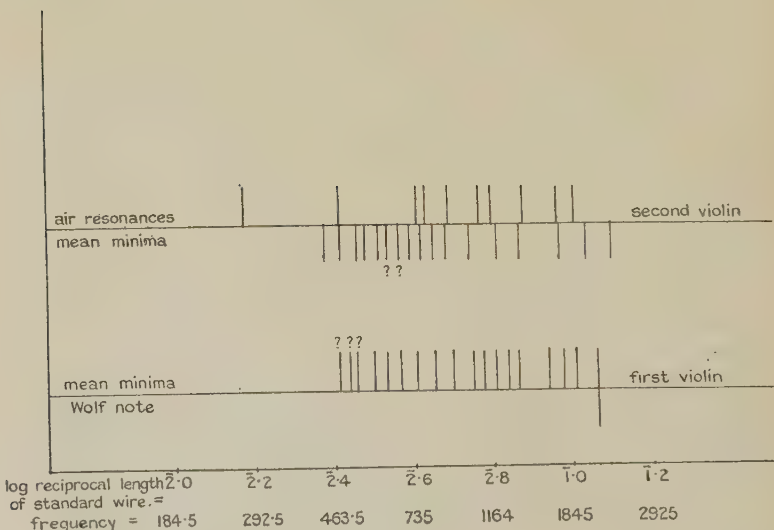
* 'Sensations of Tone,' English translation by Ellis (1875), ch. v. p. 137.

There were several marked resonances at which the note sang out fairly loudly, and the two lowest of these were at about the pitches recorded by Helmholtz.

On fig. 2 are set out the mean data obtained for the two violins examined. No wolf-note was found on the second violin, and the air resonances were not taken for the first.

It will be seen that for the second violin, two of the minima coincide with air resonances, and most of the other air resonances lie fairly close to the nearest minima. There is a very powerful air resonance about $\bar{2}{\cdot}6$, but the minimum occurs only sometimes, not always, on the graphs. The resonance about $\bar{2}{\cdot}7$ is not very marked, but the corresponding minimum appears on all the graphs.

Fig. 2.



It would seem from these results that the natural pitch of the enclosed air does affect the amplitude of the string's vibration.

The majority of the minima are probably due to natural frequencies of different parts of the violin.

It may be noted that there is coincidence between the wolf-note pitch and one of the minima on the first violin.

In order to try to trace the origin of the other minima, the bridge was loaded by small amounts. This had the

effect of shifting some minima and leaving others undisturbed; but the results found were inconsistent, and no conclusions could be drawn.

Additional Observations.

For both violins it was found that at frequencies below about 480 vibrations per sec. (1) the ordinary form of octave vibration became unstable. When started, the string vibrated in the ordinary way with a node in the middle; then the node disappeared, the string vibrated in one piece, and the fundamental was prominent in the note heard—an example of a note produced by an applied force of double its frequency. (2) When the second overtone was sounding, it was found that the wire had two pitches of maximum response close to one another instead of one, when the frequency of the impressed force was gradually altered in the neighbourhood of the wire's natural frequency, keeping the tension the same.

These two effects were in general noted only on a very slack wire—much slacker than would be the case when playing the violin.

When the third overtone was sounding at frequency 982 and a little on either side of it on the first violin, it was found that the ordinary form of vibration became unstable, and the nodes at $\frac{1}{4}l$, $\frac{1}{2}l$, $\frac{3}{4}l$ disappeared. There is a minimum on some of the curves at about this pitch.

On the first violin, places of double maximum response were found between frequencies 755–835, usually when the fundamental was sounding; sometimes this effect appeared slightly in the first and second overtones. The mean pitch corresponds to a good minimum on most of the graphs. Places of double maximum response also occurred sometimes at some pitches on the other violin.

These last observations recall the behaviour of the bar and string in cases where there was coincidence between like modes of vibration. There it was found that in each pair the pitches of maximum response kept one above and one below the natural note of the bar. In the violin case the pitch midway between the pair rose along with them, as if the proper pitch of the violin regarded as a bar were rising as the tension of the string increased. On the violins the places of double maximum response only occurred occasionally, and could not always be found again at will.

Summary.

A method of finding resonance pitches of a violin is described. Some of these correspond to natural frequencies of the air enclosed in the violin. The others are taken to correspond to natural frequencies of the body. The results of the harmonic analysis of a number of vibration photos taken at different pitches for the same point of the wire support the view that the resonances found are functions of the violin itself.

It is evident that there are a large number of resonance pitches on a violin which do not correspond to a wolf-note.

The wolf-note on the "wolfy" violin agrees in pitch with one of the resonances found.

I wish to express my grateful thanks to Professor Morton for his interest and his helpful suggestions in connexion with the above piece of work.

XIV. *The Intensities of Forbidden Multiplets. By* James Taylor, *D.Sc.* (*Utrecht*), *Ph.D.*, *The Physical Institute of the University of Utrecht* *.

Introduction.

The problem of the intensity relations of forbidden multiplets and the relative intensity of forbidden to non-forbidden lines has been studied recently at this Institute. Certain results have been put forward by Ornstein and Burger in a recent preliminary note †.

It is well known that the rule according to which the azimuthal quantum number only changes by ± 1 is frequently broken. This raises two questions. Does the Sum rule and intensity formula apply in such cases just as in normal multiplets? Secondly, how do the intensity relations of the forbidden to the non-forbidden lines depend upon conditions?

These questions were examined for the case of Cd multiplets, and yielded interesting results which we shall refer to later.

The present work contains some measurements of the intensities of two of the forbidden lines in the S–D series

* Communicated by Prof. L. S. Ornstein.

† Ornstein and Burger, *Die Naturwissenschaften J.* xv. p. 32.

of potassium, $1^2S-3^2D_{5/2}$, $1^2S-3^2D_{3/2}$ ($\lambda=4643{\cdot}172$ and $4641{\cdot}585$). They are separated by about one-half of one ängström.

Experimental Method.

Light-source and Apparatus.

A mixture of dried potassium carbonate and carbon powder, in the required proportion, was ground down so as to be as homogeneous as possible, and packed into a boring (4 mm. diameter) in an arc carbon (9 mm. diameter). The carbon was used as positive pole of an arc, the image of the middle part of which was projected upon the slit of a Rowland grating apparatus (30,000 lines per inch; radius 1 metre) which clearly resolved the lines when they were not unduly broadened.

The spectra were photographed, an iron arc spectrum being used for comparison purposes.

In other experiments a mercury-vapour lamp containing potassium was used, whilst in others electrodeless discharges in potassium vapour were employed.

Method of Intensity Measurements.

The line-intensities were measured by the photographic method of comparison *.

In the case of the forbidden lines under consideration the method is comparatively simple, for the sensitivity of the photographic plate is precisely the same for both lines, and the problem is simply to transfer blacknesses or densities of the plate into intensities. The photographic densities are measured by means of the registering microphotometer of Moll, which, in principle, is extremely simple. The image of the filament of a 10-volt lamp run on constant voltage (or of a slit placed before the lamp) is focussed upon the gelatine layer of the photographic plate, which is held in a mechanically-travelling holder in such a way that the lines of the spectrum to be measured pass successively into the correct position. The transmitted light is focussed by a lens system upon the slit of a Moll thermopile which is connected to a galvanometer. The system must, of course, be focussed for the infra-red radiation.

The deflexion of the galvanometer is registered photographically on a rotating drum, by means of which an exact record of the densities of the spectral lines is obtained.

We further require to know the density values in terms

* Ornstein, Proc. Phys. Soc. Lond. xxxvii. v. p. 334 (1925).

of light-intensity equivalents. This is done in the following manner. The image of a standard lamp run on constant voltage is focussed, so as to give a homogeneous not quite sharp image upon the slit of a spectrograph, and a series of exposures at constant current and time of exposure but with different slit-widths (*e. g.* 100, 60, 40, 26, 16, 6) is made. This plate is measured with the photometer at the wave-length which is being investigated, and gives the blackenings corresponding to 100, 60, 40, 26, 16, etc. if the slit-widths are as given above. (The intensity is proportional to the slit-width.)

If a sufficient density range is not obtained by such a series, then the current of the standard lamp is adjusted to another value and a further set is taken. The logarithms of the transmissions $\left[\log\left(\frac{\text{deflexion for clear plate}}{\text{deflexion for line}}\right)\right]$, or alternatively of the densities, is plotted against the logarithm of the slit-width. The curves for different standard-lamp currents are parallel and can be brought upon one graph by lateral displacement. When such a curve is constructed, the densities can be immediately transferred into intensity units and the relative intensities compared.

In the experiments to be described, it was more convenient to obtain the density marks (curves for intensity) with a spectrograph, for there were difficulties in utilizing the grating with different slit-widths (diffraction etc.) for such a purpose.

Initially, the procedure of cutting a plate into two sections and using one for the density curves, the other for the spectral lines, was adopted. This precaution, however, proved to be unnecessary, for it was found that plates from the same box or even plates with the same emulsion number gave almost identical results.

In comparing the intensity of the forbidden with the non-forbidden lines, the above procedure must be somewhat modified because the wave-length separation is considerable (6000 Å.) and the sensitivity of the plate is not the same. The density-mark curves are constructed for the wave-lengths which are being compared, and the relative intensity relation for the two curves is derived from a knowledge of the energy distribution in the spectrum of the standard lamp. It is found as a rule that the density curves for even widely-separated spectral regions are parallel. This is extremely convenient.

Results.

Relative Intensities of the Forbidden Lines.

Table I. gives the results obtained for the relative intensities of the forbidden lines for different concentrations of the potassium-carbonate mixture, and arc currents of about 3 amperes (240 volts).

Table I.

Ratio $\frac{\text{Pot. Carbonate}}{\text{Carbon}}$ for filling mixture.	Ratio of Intensities $\frac{1^2S - 3^2D_{5/2}}{1^2S - 3^2D_{3/2}}$.	
1	9/4	First determination.
1/2	9/5·3	First determination.
1/5	9/6	First determination.
1/2	9/6	Second determination; spectral background not so large as in first determination.

We should expect to obtain the ratio 3 : 2; but since the lower levels of the lines are normal ones, this actual ratio will be altered by self-absorption, and we should then only obtain the 3 : 2 ratio at infinitely dilute K-concentration. Unfortunately, however, the spectral background on the plates was considerable, and increased relatively with the smaller concentrations. This introduces a large experimental error, for, in the procedure whereby the line-intensity is measured, the integrated intensity corresponding to the blackening produced by the line and the background must be found, and from this must be subtracted the corresponding intensity of the background.

At higher concentrations of the potassium carbonate it was also found that the lines were broadened. Nevertheless, the results obtained are good and show the correct limit, so that we may conclude that the ratio of the intensities is certainly for dilute concentrations that given by the Sum rule; and, if the results of Ornstein and Burger * upon other cases are taken into consideration, we conclude with a fair degree of probability that the ratio is normal for forbidden multiplets.

Further experiments were carried out with a view to reduce

* *Id., ibid.*; also v. Alphen, Dissertation, Utrecht, 1927.

the relative intensity of the continuous background. In a mercury arc containing about 1 per cent. of potassium the lines were considerably broadened, and were not resolved into the two components.

On the other hand, with electrodeless discharges in potassium vapour at various pressures (different temperatures) it was not found possible to obtain the forbidden lines at all. This is also in accord with the results of Ornstein and Burger * for cadmium.

Relative Intensities of the Forbidden to the Non-forbidden Lines.

A series of experiments were carried out to investigate the relative change of the ratio of the maximum intensity of the compounded 4642 lines to the non-forbidden lines 4044 and 4047, which have the same lower level as the forbidden lines ($1^2S-3^2P_{3/2}$, $1^2S-3^2P_{1/2}$).

The ratio is much too small to measure in the normal way, but the difficulty was overcome by comparing the lines 4642 with one of the ghosts of the non-forbidden lines given by the grating. The ghosts were four in number, two from each line, and the intensities were about 1 per cent. of the line-intensities. As a rule the first ghost on the short wave-length side was used; but it was determined initially by experiment that the ratio of the lines and ghosts was constant, so that any of the ghosts could be used for measurements.

For our purpose, only the relative change of the intensities under different conditions is required, so that it is unnecessary to bring the ratios expressing the intensities to real values.

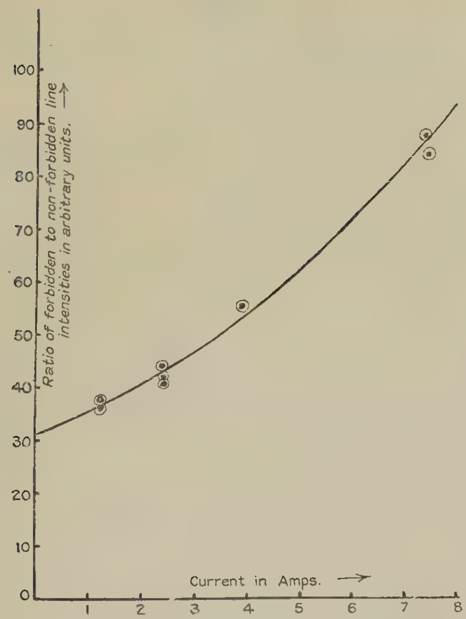
The results of measurements at different current densities are shown in graph I. It is to be seen that the ratio depends upon the intensity of the arc current, the intenser currents giving a greater relative intensity, as is to be expected. The trend of the graphs is similar to that found for the cadmium arc; but it must be remembered that the present case is much more complicated because of the dilution by carbon.

Graph II. shows the variation of the ratio with concentration of potassium carbonate in the filling mixture. The ratio increases as the potassium content of the mixture increases, whilst at the same time the lines become broadened. This result is also in a general way to be expected from the fact of the increased potassium-ion concentration in the arc.

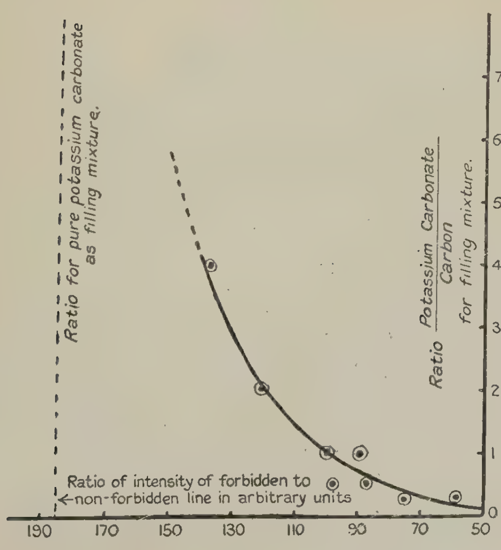
Experiments with an arc in air at different pressures gave inconclusive results. The air lines came up very strongly at pressures lower than abont 8 cm., and in a general way the ratio of the forbidden to the non-forbidden lines was found to become smaller at the lower pressures.

* Ornstein and Burger, *Die Naturwissenschaften J.* xv. p. 32.

GRAPH I.



GRAPH II.



It cannot, however, be claimed that the results were definite, for divergent results were frequently found under the same conditions. This arises probably from "spurts" of potassium-salt vapour and local increases of the ionic concentration.

Intensity of Forbidden Multiplets etc.

The experiments of Ornstein and Burger * upon cadmium were carried out on some of the $p-f$ lines, employing an arc between cadmium electrodes.

The relation of the lines p_0-f, p_1-f, p_2-f ($\lambda=2961$, 2862, 2891) was within the limits of experimental error, 1 : 3 : 5, which is to be expected from the Sum rule, as the f-term is not resolved. The second multiplet yielded similar results.

The intensities of a $p-f$ multiplet can also be represented by a number given by the ratio of a pf line to the adjacent corresponding $p-d$ line with the same total and the inner quantum number (cp. above). To investigate the problem from this point of view, an electrodeless discharge in pure cadmium vapour was first investigated, but though the $p-d$ lines were intense, no trace of the pf lines was observed.

The cadmium arc in air at different pressures was then examined, and it was found that the ratio $pf:pd$ was a function of the pressure approximately proportional to $p^{0\cdot4}$.

It was also found, at constant pressure, that the ratio $pf:pd$ increased with increasing current. A corresponding increase in the width of the pf lines was also observed.

Conclusions.

1. The Sum rule is valid for forbidden multiplets.

2. The relative intensity of the forbidden to the non-forbidden lines increases with increasing current and vapour density. This is in accord with the hypothesis that the forbidden lines are brought up by the action of ionic electric fields and that they increase in intensity with increase of the fields. This explanation is in accordance with the results of Takamine and Werner †.

It is a great pleasure to me to express my gratitude to Prof. Ornstein, who suggested the problem, and to Dr. v. Cittert and Dr. Burger for their continued help and advice. I am also indebted to the International Education Board for the Fellowship which enabled the work to be carried out, and to Mr. v. Hasselt, who helped with many of the measurements.

September 29th, 1927.

* Ornstein and Burger, *Die Naturwissenschaften J.* xv. p. 32.
† Takamine and Werner, *Die Naturwissenschaften J.* 1925.

XV. *The Classification of Metallic Substances.*

By W. HUME-ROTHERY, *M.A., Ph.D.**

IN recent years considerable discussion has taken place as to whether the different constituents to be found in metallic alloys are to be classified as mixtures or as compounds. From some points of view it may be said that the matter is of little importance, for it may be argued that the properties of a substance do not depend on the particular label which it is given. But as a position has arisen in which different writers are using the term "compound" with entirely different meanings, it may be well to examine the matter a little more closely, and to see if a satisfactory scheme of classification can be devised. The problem is one which does not concern metallurgy alone, but which may arise in many branches of science when the structures of solid substances have been examined in more detail.

The usual text-book definitions for distinguishing between compounds and mixtures give the following three tests for a definite chemical compound:—(1) Fixity of composition; (2) decomposition into its constituents by chemical means only, and not by physical or mechanical means; (3) the possession of properties which, unlike those of a mixture, cannot be calculated from the properties and proportions of the constituents.

The last two of these criteria are admittedly unsatisfactory, and the generally-accepted test for a compound is that it should contain the elements in proportions corresponding to a possible formula—assuming that the atomic weights are known—and that these proportions should be independent (within limits) of the conditions of formation, as for example the composition of the liquid from which the solid phase separates.

In the majority of cases the differences are so clear that little difficulty arises, but the above criteria are really by no means infallible. It has, for example, been pointed out by U. R. Evans (Trans. Faraday Society, October 1923) that some of the metallic oxides, such as those of iron, tungsten, and molybdenum, which are generally admitted to be compounds, are, in fact, of distinctly variable compositions, so that the law of constant proportions appears to break down. Even such a well-defined compound as ferric oxide can form solid solutions of limited extent

* Communicated by Dr. N. V. Sidgwick, F.R.S.

with its component elements, so that perfectly homogeneous samples of different compositions can be obtained; but few chemists would refuse to admit it as a compound on this ground.

It must, I think, be admitted frankly that the ordinary means described for distinguishing between compounds and mixtures have failed to consider the possibility of a compound forming a solid solution with its component elements, and the way in which such solid solutions are to be distinguished from those in which no compounds are formed.

From the point of view of the older atomic theory, however, the distinction was quite plain. The physical mixture of the two elements was considered as a mixture of the individual atoms or molecules, whilst in the compound the different atoms were bound together into compound molecules, and these behaved as distinct entities. For many years this conception was purely hypothetical, but the comparatively recent work on positive-ray analysis and mass spectra has led to the conclusive experimental detection of the individual molecule, the existence of which is now a fact and no longer merely a supposition. The formation of these molecules is regarded as due to the sharing or exchange of electrons between two atoms. At first sight, therefore, it might seem justifiable to define a metallic compound as one in which the constituent atoms had united to form a compound molecule, but unfortunately almost all the methods for detecting the existence of individual molecules depend upon the examination of the substance in the gaseous or liquid state, which is in general very difficult at the temperatures of molten metals. A start in this direction has been made by the Japanese scientists, notably H. Honda and his collaborators, by the study of the magnetic properties of molten alloys, and in this way it has been shown that some of the phases of intermediate composition found in solid alloys correspond to entities in the liquid, whilst others do not. In another connexion the work of Barratt and others on the spectra of metallic vapours has led to the observation of band spectra, which are usually ascribed to compound molecules and not to free atoms. But even in cases such as these, there is always the possibility that a true compound may be formed in the solid, but may decompose on melting or volatilisation. In general, therefore, we have to examine the solid alloys alone, and here the connexion between the molecule and the crystal structure is not always clear. In many compounds, notably those met with in organic chemistry, the molecules retain their

individuality in the solid crystal, but in others, such as sodium chloride, there is no sign of the individual molecule in the crystal lattice, although the high-temperature work of Nernst and others shows that, in the gaseous state, sodium chloride consists entirely of compound molecules. We are not, however, driven to believe that sodium chloride is a compound in the gaseous state and a mixture when solid, because the work of Rubens on Reststrahlen indicates conclusively that the units in the solid crystal are not neutral atoms of sodium and chlorine, but are charged ions, each sodium atom having lost and each chlorine atom gained an electron. This is in agreement with the more refined methods of X-ray analysis. It must again be emphasized that this is now a definite experimental fact, and not merely a supposition.

The above considerations would therefore seem to indicate how we may best classify the various intermediate phases which are so often met with in alloys, and it is suggested that the following scheme of classification might well be adopted :—

At the head of our classification of solid metallic phases we may place the METALLIC ELEMENTS, and we may note that these may be isotopically simple or isotopically complex.

We may next define as PRIMARY METALLIC SOLID SOLUTIONS those solid solutions in which the crystal structure of the parent metal is retained. These primary solid solutions form the "end phases" when the equilibrium diagrams are drawn in their usual form. The X-ray spectrometer shows that these are of two types—the substitutional type, in which the solute atoms replace the atoms of the solvent metal upon its lattice, and the interstitial type, in which the solute atoms fit into the spaces in the lattice of the parent metal. In the case of ternary alloys the primary solid solution may be of both types: thus in an austenitic manganese steel we have a primary substitutional solid solution of iron and manganese forming an interstitial solid solution of carbon.

We may further define as SECONDARY SOLID SOLUTIONS those solid solutions of intermediate composition in which the crystal structure is different from that of the parent metal, but in which there is no indication of the formation of compound molecules or that any exchange or transference of electrons is taking place.

And, finally, we may define as INTERMETALLIC COMPOUNDS OF FIXED OR VARIABLE COMPOSITION those intermediate

phases where there is evidence, direct or indirect, that electron exchange or transference is taking place or that a compound molecule is being formed.

The above scheme would appear to give a satisfactory fundamental distinction, although it must be admitted frankly that in some cases our experimental methods may not yet be sufficiently developed to distinguish between the secondary solid solutions and the intermetallic compounds. In such cases we must just select the more probable alternative. Where, for example, we have a phase of fixed or practically fixed composition, or one which shows markedly different properties from those of its components, we may reasonably classify it as an intermetallic compound, even though we cannot definitely prove that electron transference or sharing has taken place or that a compound molecule has been formed. On the other hand, where we have a phase of variable composition whose properties resemble those of its constituents we may reasonably call it a secondary solid solution. It has, for example, been suggested by W. Rosenhain (J. Inst. Metals, xxxv. p. 353, 1926) that in cases such as those of the β brasses we have to do with what is essentially a solid solution of zinc in an allotropic form of copper. In many ways this illustration is very apt, for it indicates just what it is desired to express above, namely that although the lattice type of the β brasses is different from that of copper, the atoms have not shared or exchanged electrons, but are bound together by forces of the same nature as those in the primary solid solutions. In order to avoid confusion, it is thought better to use the term secondary solid solution. For where a metal, such as iron, exists in different allotropic forms, each allotropic form may give rise to its own primary solid solution, and the term allotrope is better confined to this phenomenon.

In conclusion, it may be well to refer very briefly to two previous suggestions that have been made in connexion with the subject of this paper.

It was long ago suggested by C. A. Edwards that in complex alloy systems every phase except the two end phases, which we have called primary solid solutions, should be regarded as based on a definite compound. It would seem that this is using the term compound in a sense very different from that of the chemist. For it ignores the possibility that the atoms of two elements may build themselves into different lattice patterns without any electron transference or sharing taking place.

A quite different suggestion has recently been made by

Westgren and Phragmén (Phil. Mag. vol. l. p. 311, 1925), according to whom in an ideal chemical compound structurally equivalent atoms are chemically identical, whilst in an ideal solid solution all atoms are structurally equivalent. Thus, in the case of the β brass solid solution, which ranges round the equi-atomic composition, the above authors would call it a solid solution in a compound if the equi-atomic alloy had the zinc and the copper atoms arranged in a definite pattern relatively to each other, and they would call it a simple solid solution if the two kinds of atoms were arranged at random on a common lattice. It would again seem that this is using the term chemical compound in a new sense. For whilst it is quite true that in all definite compounds structurally equivalent atoms are chemically identical, the reverse argument does not necessarily follow. If, for example, we imagine two sets of spheres of different sizes to exert attractive forces, we can well understand how some whole-number ratios may give a particularly close packing when those of one size occupy definite positions relatively to those of the other. But this may occur without the exertion of any forces other than those which exist when one set of spheres alone is packed together, and it is only in some cases, the so-called "Riesenmolekül" substances, that these crystal forces are regarded as forces of chemical combination. In the same way, the fact that a solid solution of wide range shows a maximum hardness, melting-point, or electrical conductivity at some simple whole-number ratio of atoms does not necessarily imply that a compound is formed. It may mean that a definite compound exists, or it may simply mean a symmetrical packing of the atoms, and the term chemical combination, in the chemist's sense of the word, implies more than the mere arrangement of atoms in symmetrical patterns. A term such as a SYMMETRICALLY-PACKED SOLID SOLUTION can readily be devised for this phenomenon. But it is desired to emphasize that a phenomenon of this sort may be quite distinct from chemical combination in the usual sense of the word, and as the term "chemical compound" has already a clear and definite meaning to the chemist, it would seem most undesirable that the physicist should use the same term with an entirely different meaning.

From some points of view it may be said that any scheme of classification is arbitrary, and is to some extent a mere matter of definition, but it is suggested that the system described above is one in which the fundamental distinctions

are clear, and in which no term has been given a meaning different from that with which it is usually associated. It can only lead to confusion if terms such as "chemical compound" are used loosely, and are given different meanings in different branches of science, and if the present paper helps to remove some of the existing confusion, it will have served its purpose.

The author must express his thanks to Dr. N. V. Sidgwick, F.R.S., for his kind interest and criticism.

Magdalen College, Oxford.

XVI. *Electrodeless Discharges.* *By* J. S. Townsend, *M.A., Wykeham Professor of Physics, Oxford, and* R. H. Donaldson, *M.A., University College, Oxford* *.

1. THE improvements in the methods of maintaining continuous electric oscillations by means of valves may be used in experiments on electrodeless discharges in gases, and the properties of the discharges may thus be determined to a degree of accuracy which cannot be attained with damped oscillations obtained by means of disruptive discharges.

Several methods of using valves in short-wave generators are given by Mesny in his book, 'Les Ondes Électriques Courtes,' where an account is also given of the first experiments made by Gutton with gases at low pressures rendered conducting by the action of continuous oscillations.

Continuous wave generators have been found very useful for many ordinary experiments, also in connexion with researches when it is desirable to have a simple means of testing samples of gases. Since the high-frequency currents may be produced in electrodeless tubes, the effect of impurities from metal electrodes in a gas may be avoided, and in many cases a discharge is more easily obtained with external electrodes than with electrodes in contact with the gas.

With ordinary cylindrical tubes the discharge is most easily obtained when the direction of the electric force is along the axis of the tube, and tubes without electrodes may

* Communicated by the Authors.

be used to show the direction and the intensity of electric forces in the field of an oscillating circuit.

The tubes may also be used with external electrodes in the form of sleeves, and the potentials required to produce a discharge with the sleeves at various distances apart may be found. Thus a long tube with a sliding sleeve may be calibrated and used as a voltmeter to determine the amplitude of a high-frequency potential.

In this paper we propose to describe some of the properties of electrodeless discharges which we have determined, using tubes that had been made up at various times in the laboratory to examine the spectrum of samples of gases. These tubes were of various shapes, the simplest forms being cylindrical or spherical. It is generally of advantage to use cylindrical tubes not less than 3 or 4 cm. in diameter and large spheres 10 or 12 cm. in diameter.

In another paper we propose to describe experiments with tubes of different shapes specially designed to investigate properties of high-frequency discharges.

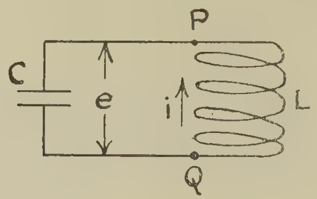
2. In considering the forces in the field inside or outside a solenoid, it is necessary to distinguish between the electric forces due to electric charges on the solenoid and those due to the variation of the magnetic force. The latter will be indicated by the suffix m; thus $\mathrm{E}m$ and $\mathrm{F}m$ will be used to denote the electromotive force round a circle, and the electric force along the tangent to the circle due to electromagnetic induction, and E and F the potentials and force due to electric charges. In the field either inside or outside a solenoid the forces F are generally very large compared with the forces $\mathrm{F}m$. At points of the central plane—the plane through the centre of the solenoid with the axis normal to the plane—the forces F are parallel to the axis, and the forces $\mathrm{F}m$ in directions perpendicular to the axis.

3. In order to determine the relative values of these forces the simple case of a circuit consisting of a solenoid of self-induction L and a condenser of capacity C may be considered, and for the purpose of an elementary calculation it may be assumed that the capacity of the condenser is large compared with the capacity of the coil. In the principal mode of oscillation the current at any instant is approximately the same in all the turns of the solenoid. Let $i = \mathrm{I} \sin pt$ be the current, and $e = \mathrm{E} \cos pt$ the potential difference between the points P and Q (fig. 1) at the ends of the

N 2

solenoid when continuous oscillations are maintained. If n be the number of turns per unit length of the solenoid, l the distance between P and Q, the magnetic force H inside the

Fig. 1.



solenoid is $4\pi ni$, so that e, which is equal to $\mathrm{L}di/dt$, may be expressed in terms of H. Thus

$$e = \mathrm{L}\frac{di}{dt} = \pi na^2 l\frac{d\mathrm{H}}{dt},$$

and the electric force F on the line from P to Q is

$$\mathrm{F} = \frac{e}{l} = \pi na^2\frac{d\mathrm{H}}{dt},$$

$2a$ being the diameter of the solenoid.

The force Fm inside the solenoid is obtained from the electromotive force Em round a circle of radius b in the principal plane. Thus

$$\mathrm{F}m = \frac{\mathrm{E}m}{2\pi b} = \frac{b}{2}\frac{d\mathrm{H}}{dt},$$

and the ratio of the two forces is

$$\mathrm{F}/\mathrm{F}m = 2\pi na^2/b.$$

Hence in a solenoid, when the distance between the turns is not greater than 1 cm. ($n > 1$) and the radius of the solenoid not less than 5 cm., the electric force F is more than thirty times as great as the electromagnetic force Fm.

It is remarkable that in the explanation usually given of the currents in electrodeless tubes or bulbs placed inside solenoids the large forces F are not taken into consideration. It is generally supposed that the currents in the gas flow in circles round the axis under the action of the forces Fm as in the theory recently given by Sir J. J. Thomson *.

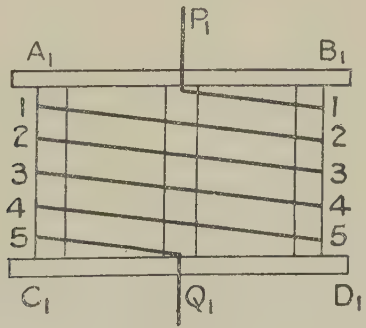
* Sir J. J. Thomson, Phil. Mag. Nov. 1927.

4. In general the capacity of the coils cannot be neglected when the capacity of the condenser is reduced in order to obtain short-waved oscillations; and with short solenoids the ordinary formulæ for H and L are inaccurate. With a small condenser the current flowing through the turns of the wire near the centre of the solenoid may be much greater than the current flowing into the condenser.

In these cases it is necessary to determine the forces in the field experimentally and to find the wave-lengths of the oscillations by means of a short-wave wave-meter. The wave-meters used in these experiments were calibrated by adjusting the wave-length of a generator to an exact multiple of a 5-metre or 10-metre wave which can be measured by means of Lecher wires*.

5. In order to find the relative values of the forces F and Fm, and to show how the effects of the forces F may be exhibited by means of a large bulb placed inside a solenoid, two solenoids G_1 and G_2 were used. The solenoids were wound differently so that when currents giving the same values of the force Fm in the central plane flow in each coil, there is a large difference in the intensity and in the distribution of the forces F. The solenoids had the same number of turns and were of the same section and same length. They were wound in hexagonal form on frames consisting of six ebonite rods 10 cm. long fixed at equal distances apart to two wooden rings, as shown in figs. 2 and 3.

Fig. 2.



Solenoid G_1.

The outer diameter of the rings was 19 cm. and the inner diameter 15·5 cm. In the diagram (fig. 2) A_1B_1 and C_1D_1

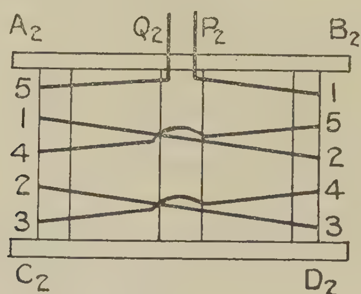
* J. S. Townsend and J. H. Morrell, Phil. Mag. xlii. p. 265 (1912).

are the rings of the frame of the coil G_1, and A_2B_2 and C_2D_2 those of the coil G_2 in fig. 3.

The coil G_1 was wound in the ordinary manner in a single layer of 13 turns, 7 mm. apart, from the upper to the lower part of the frame.

The coil G_2 was wound with $6\frac{1}{2}$ turns, 14 mm. apart from the upper to the lower part of the frame, and $6\frac{1}{2}$ turns from the lower to the upper part, also at 14 mm. apart. Thus the two coils were each 8·4 cm. in length.

Fig. 3.



Solenoid G_2.

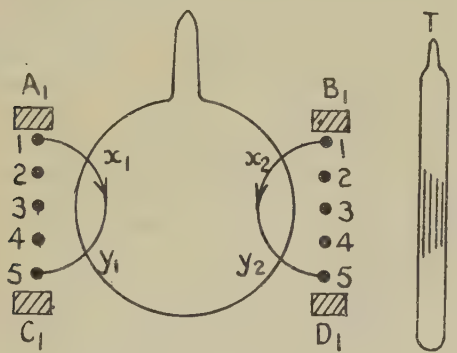
In the diagrams the coils are represented as having five turns each instead of thirteen in order to simplify the figures. The centre of the third turn (assuming there were five turns in the solenoid) may be taken as being at zero potential for the high-frequency oscillations; and as the potential difference between the ends is $C = E \cos pt$, one end oscillates in potential from $+E/2$ to $-E/2$, while the other end oscillates from $-E/2$ to $+E/2$. The average rate of change of potential along the wire is $e/2\pi Na$, and the potential difference between adjacent points (1 and 2, 2 and 3, etc.) of consecutive turns is e/N. Thus the potential difference between the points 1 and 5 (marked at the sides of the coils in figures) is $4e/N$, and between the points 2 and 5, $3e/N$, etc.

6. In the coil G_1 the turns which differ in potential by the largest amount are far apart, 8·4 cm., so that there is a large electric force in the field at a considerable distance from the coil, both inside and outside. The lines of force inside the coil are indicated by the arrows in fig. 4, which represents a section of the coil through its axis, the vertical rows of points on each side being the sections of the wire.

When a spherical bulb containing gas at a low pressure is

placed inside the solenoid G_1, there are large forces along the lines from x_1 to y_1 and from x_2 to y_2 which cause a discharge in the gas when the current in the solenoid is increased up

Fig. 4.

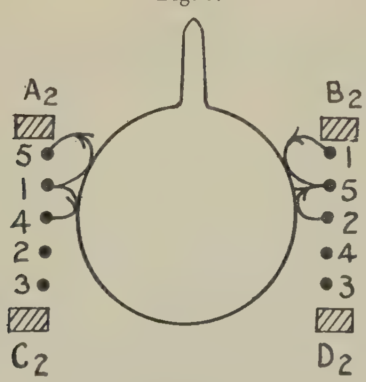


Solenoid G_1.

to a certain point. The glow which accompanies the discharge is equally distributed above and below the central plane.

7. In the coil G_2, which is shown in section in fig. 5, the turns of the wire which differ most in potential are near together (7 mm. apart) in the upper part of the solenoid.

Fig. 5.



Solenoid G_1.

The electric force is therefore much greater in the upper part of the field above the central plane than in the lower part of the field. The lines marked with arrows in fig. 5 indicate the distribution of the force F in the coil G_2.

In this case, when the bulb containing the gas at low pressure is placed inside the solenoid as in fig. 5 with the centre of the bulb at the centre of the solenoid, there is no current in the gas when the current in the solenoid is the same as that required to produce a discharge with the bulb in a similar position inside the solenoid G_1. In order to obtain a discharge, the current which is required with the bulb inside G_2 is more than four times the current required with the bulb inside G_1.

When the discharge is obtained with the bulb inside the solenoid G_2, the glow is concentrated in the upper part of the bulb where the electric force is large. Also, by raising the bulb so as to bring more of the gas into the space where the force is large, the discharge is obtained with a smaller current in the solenoid.

8. When a cylindrical tube T containing a gas at low pressures is placed outside the solenoid G_1 in the position shown in fig. 4, a discharge is produced in that part of the tube where the lines of force are parallel to the axis of the tube.

The discharge may be started by holding the tube close to the solenoid, and if the tube is withdrawn keeping it parallel to the axis of the solenoid, the length that is illuminated contracts.

The shading of the centre of the tube T in fig. 4 indicates the extent of the glow when the tube is moved away from the solenoid.

If the tube is rotated and its axis brought into the central plane of the solenoid, the discharge ceases even when the tube is quite close to the coil.

9. In most of the experiments which we describe the wave-lengths of the oscillations were from 100 to 40 metres, and the capacities of the coils were comparable with the capacities of the condensers, so that it was necessary to measure the electric potentials and the electromotive forces due to induction by direct experiment.

The potential difference E between the ends of the solenoid (fig. 1) may be obtained accurately by the following method. A system of three adjustable condensers is substituted for the capacity of C, two in series each of capacity 2S and one in parallel of capacity C' as shown in fig. 6.

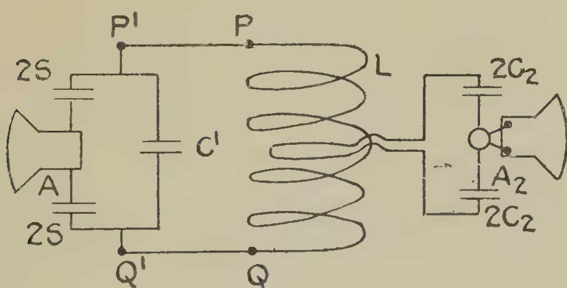
The condensers 2S are connected through a low-resistance milliammeter A. The capacity C between the terminals P' and Q' is $C'+S$, and it may be adjusted to any required value by adjusting C'.

The ammeter current depends on the capacities 2S, and a value of S may be selected to obtain an ammeter deflexion in the sensitive part of the scale.

The condensers 2S were carefully constructed, and a range of small capacities was determined very accurately. They were calibrated by comparison with a small standard condenser at intervals of the scales over a range from 2S = 15 cm. to 2S = 120 cm.

In the arrangement shown in fig. 6 the milliammeter is connected in the circuit at a point of zero potential, and the readings are not affected by the capacity of the milliammeter.

Fig. 6.



If I be the current in the milliammeter, the potential difference E between P′ and Q′ is I/Sp.

The electromotive force E*m* was measured in a similar manner. A loop of copper wire 12·5 cm. in diameter was fixed in the central plane of the solenoid, and the ends brought out to a pair of adjustable condensers which were connected in series through a thermojunction. The current I_2 in the condensers was obtained from the deflexion of the microammeter A_2.

If $2C_2$ be the capacity of each condenser, l_2 the self-induction, and R_2 the resistance of the loop circuit, the quantities l_2p and R_2 were small compared with I/C_2p. Hence the electromotive force E*m* is equal to I_2/C_2p.

Since the current I_2 is proportional to C_2, the deflexion of the microammeter may in this case also be brought to any convenient position on the scale by adjusting the value of C_2.

With oscillations of 83 metres in wave-length the ratio E : E*m* was 30 : 1, the same for each of the coils G_1 and G_2.

10. The experiments on the discharges in bulbs were made with two spherical bulbs of 12·5 cm. in diameter, one

containing neon at 1·06 mm. pressure and the other neon at ·1 mm. pressure.

In one set of experiments the coils G_1 and G_2 were connected to valves with the two condensers 2S connected in series through an ammeter as shown in fig. 6.

The oscillatory current I in the condensers was gradually increased by increasing the heating current in the filament. At a definite point I_1 a discharge is obtained in the bulb, and the potential difference $E_1 = I_1/Sp$ between the ends of the solenoid required to start the discharge is thus obtained.

If the current in the solenoid be reduced after the discharge is started, the current in the gas diminishes at first uniformly with the current in the solenoid, but at a certain point $I = I_2$ the current in the gas ceases abruptly. The minimum potential $E_2 = I_2/Sp$ under which the current continues to flow in the gas is less than one-third of the minimum potential E_1 required to start the discharge.

The amplitudes of the oscillatory potentials are obtained from the observed mean potentials by multiplying the latter by the factor $\sqrt{2}$.

The critical potentials E_1 and E_2 for the two solenoids, for discharges in the bulb containing neon at 1·06 mm. pressure in the position shown in figs. 4 and 5, are given in Table I., the wave-length of the oscillations being 83 metres in all cases.

The numbers in the table are the amplitudes of the potentials in volts.

The corresponding values of the electromotive forces Em_1 and Em_2 round a circle of 12·5 cm. diameter are also given.

TABLE I.

	E_1.	E_2.	Em_1.	Em_2.
Solenoid G_1	280	88	9·4	2·9
Solenoid G_2	1344	372	44	12·3

The amplitude of the force Fm in the central plane is $Em/40$, the highest value being that obtained in the solenoid G_2. The mean value in this force, $2Fm/\pi$, is ·7 volt per centimetre, which is too small a force to impart sufficient energy to electrons in neon at 1·06 mm. pressure to obtain any appreciable effect of ionization by collision.

The discharges in the gas must therefore be attributed to the potentials E; but it is difficult to obtain the electric force F from these experiments where the conductors are so far from the bulb containing the gas. A closer estimate of the critical electric forces in the gas is obtained from the experiments in which the oscillation potential is applied to external electrodes resting on the surface of the bulb.

The experiments with the tube containing neon at ·1 mm. pressure gave results which were nearly the same as those obtained with the gas at 1·06 mm. pressure.

11. Experiments were also made with the same bulbs under the action of forces in a secondary circuit loosely coupled to a high-frequency generator maintained in oscillation by a valve, the wave-length of the oscillations being the same as in the experiments described above. Any effects of the harmonic oscillations which occur when a solenoid is directly coupled to a valve are greatly reduced in the secondary circuit.

The secondary circuit which was used consisted of the solenoid G_1 connected to the condenser system (SC′) shown in fig. 6. The capacity C of the system was adjusted to a point just above the value corresponding to resonance. in order to avoid abrupt changes in potential which occur when a discharge is started with the capacity below the point of resonance. These changes are due to a small additional capacity introduced by the discharge in the gas, which causes the total capacity of the secondary circuit to approach or pass through the exact value corresponding to resonance.

The potentials E at the terminals of the solenoid G_1 are obtained from the current in the milliammeter which is between the condensers 2S. Continuous variations in the potential E of the secondary circuit were obtained by varying the filament current of the valve in the generator.

Determinations of the critical potentials were made with the bulb containing neon at 1·06 mm. pressure placed inside the solenoid in the secondary circuit, as in fig. 4, to test the results obtained with the same solenoid G_1 when it was connected to a valve.

With the bulb in the secondary circuit the values obtained for the critical potentials were $E_1 = 288$ volts and $E_2 = 85$ volts, which are practically the same as the potentials 280 and 88 obtained with the solenoid connected to the valve.

12. In order to apply the total potential $1/Sp$ more directly to the gas, two rings of lead-foil were fixed to the bulb, at 10 cm. apart, which acted as external electrodes.

The lead rings were connected by short wires to the terminals P and Q of the solenoid, and were used with the bulb either inside or outside the solenoid. The rings were split, to prevent induced current flowing round them when the bulb was inside the solenoid.

The critical potentials obtained with the bulb inside the solenoid were almost the same as those obtained with the bulb outside. In the former case the values of E_1 and E_2 were found to be 125 volts and 34 volts, and in the latter case 130 volts and 35 volts respectively. This shows that the magnetic force inside the solenoid has no marked effect on the discharge. As the electrodes were 10 cm. apart, the mean force between them corresponding to the potential E_2 is 3·5 volts per cm. This is not the mean force at points in the gas, as the potentials at the external electrodes are not the potentials at neighbouring points in the gas. The electric force in the gas is easily found from the values of E_2 obtained for cylindrical tubes with external electrodes at different distances apart.

13. The minimum potentials E_1 and E_2, required to start and to maintain discharges, were determined for gases contained in cylindrical tubes about 30 cm. long. External electrodes of lead-foil in the form of sleeves which fitted the tubes were connected to the terminals of the inductance of a secondary circuit, and gradual variations of potential were obtained by small changes in the filament current of the valve in the primary circuit. The potentials E_1 and E_2 were determined for different distances between the electrodes.

When the discharge starts a glow extends in a uniform column in the part of the tube between the electrodes and the current continues to flow as the potential is reduced. There is a minimum value E_2 of the potential under which a uniform glow is maintained in the tube.

14. In a quartz tube 4·2 cm. in diameter, containing pure neon at one millimetre pressure, the minimum potential E_2 is about one-sixth of the starting potential E_1. In this tube the glow retains its characteristic red colour as the potential is reduced to E_2.

In neon and helium small traces of impurities have an effect on the colour of the discharge, which becomes very marked as the potential is reduced. Small amounts of impurity in neon are thus easily detected even when the amount is so small that it is difficult to detect it spectroscopically. This method of testing gases has been in use for

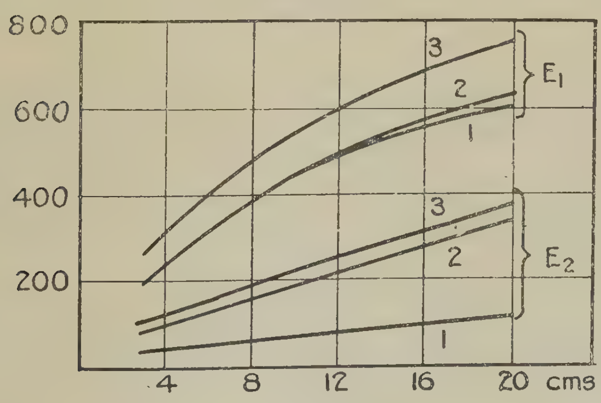
some time in the laboratory in connexion with researches for which pure gases were required.

15. In order to determine the effect of the diameter of the tube on a discharge, the critical potentials were measured for three tubes containing nitrogen at ·25 mm. pressure. The diameters of the nitrogen tubes were $2a_1 = 2{\cdot}9$ cm., $2a_2 = 3{\cdot}5$ cm., and $2a_3 = 3{\cdot}9$ cm.

The critical potentials for the tube 3·5 cm. in diameter were almost the same as those of the tube 3·9 cm. in diameter. With the small tube, 2·9 cm. in diameter, the critical potentials were larger than with the tube 3·9 cm. in diameter.

The minimum potential E_1 required to start a discharge, and the minimum E_2 required to maintain the current with the distances D between the sleeves, from D=2·5 cm. to D=20 cm., are given by the curves, fig. 7. The ordinates

Fig. 7.



represent the amplitudes of the potentials in volts and the abscissæ the distances D in centimetres. The upper set of curves give the potentials E_1 and the lower set the potentials E_2.

Curves 1 give the potentials for the tube 4·2 cm. in diameter, containing neon at one millimetre pressure.

Curves 2, the potentials for the tube 3·9 cm. in diameter, containing nitrogen at ·25 mm. pressure.

Curves 3, the potentials for the tube 2·9 cm. in diameter, containing nitrogen at the same pressure.

16. For distances between the external electrodes exceeding 5 cm. the increase in the potential E_2 is proportional to the increase in D, as shown by the straight parts of the curves. The distribution of electric force in the gas near the electrodes is independent of the distance D, and the force F in the uniform column of the glow in the central part of the tube is constant. Hence the increase in the potential E_2 corresponding to the increase $D_1 - D_2$ between the electrodes $F \times (D_1 - D_2)$.

In the neon tube E_2 increases from 66 volts to 106 volts when D is increased from 10 to 20 centimetres, hence the amplitude of the force in the gas is 4 volts per centimetre, and the average force $2F/\pi$ is 2·55 volts per centimetre.

In the nitrogen tube 3·9 cm., in diameter the minimum force F required to maintain a current is 14 volts per centimetre, and the average force $2F/\pi$ 9 volts per centimetre.

The principle of this method of finding the force in a long column of gas is the same as that used by Kirkby*, where he finds the force in the uniform positive column of a current maintained by a battery of cells by changing the distance between the electrodes in the gas.

17. The motion of the electrons in gases at pressures of the order of one millimetre in high frequency currents may be deduced from the velocities acquired in fields of uniform constant force †. Under the continued action of the force $E \sin pt$, the electrons acquire energy of agitation at the same rate as under the action of a constant force Z, equal in intensity to the mean value $2F/\pi$ of the high-frequency force.

The mean energy of agitation of electrons in the high-frequency discharge is the same as that attained in the uniform field Z, and the distance traversed by a group of electrons in each half-cycle is obtained from the velocity W in the direction of the force Z. The mean velocity of agitation U and the velocity W for neon have been determined by Bailey ‡, and the values corresponding to the force of 2·5 volts per cm. in the gas at one millimetre pressure are $U = 1{\cdot}98 \times 10^8$ and $W = 3{\cdot}3 \times 10^6$ cm. per second. Thus the mean energy of agitation is the energy acquired under a potential of about 10 volts.

The distance traversed by a group of electrons in the

* Rev. P. J. Kirkby, Phil. Mag., April 1908.

† 'Motion of the Electrons in Gases,' Clarendon Press, Oxford.

‡ V. A. Bailey, Phil. Mag. xlvii. p. 379 (1924).

direction of the force F in the half-period $T/2 = 1{\cdot}4 \times 10^{-7}$ sec. is 4·6 mm. Thus in each half-period of the oscillations the electrons acquire a small amount of energy. When they lose a large proportion of their energy in collisions with atoms of the gas they move backwards and forwards several times in the direction of the force F before they regain sufficient energy to ionize the atoms or to excite radiation.

XVII. *Waves associated with Moving Electrons.*

By Sir J. J. Thomson, *O.M., F.R.S.**

THE experiments recently made by Professor G. P. Thomson and Mr. Reid on the scattering of cathode rays by very thin sheets of metal or celluloid supply very direct evidence in favour of the view that a moving electron —even a uniformly moving one—is accompanied by waves whose wave-length depends on the velocity of the electron, the product of the wave-length and the velocity being, approximately at any rate, constant.

The existence of these waves is required by the very interesting theory of wave dynamics put forward by L. de Broglie. I hope to show in this paper that the waves are also a consequence of classical dynamics if that be combined with the view that an electric charge is not to be regarded as a point without structure, but as an assemblage of lines of force starting from the charge and stretching out into space. These constitute a system of a much more detailed and elaborate character than a point charge. When the charge is at rest and at a large distance from other charges the lines of force are uniformly distributed round the centre; the lines can, however, be displaced relatively to each other, and they are so displaced when the particle is acted upon by an electric force, such for example as that produced by the passage through the region occupied by the lines of force of a rapidly-moving electron. The displacement of one line of force relatively to another sends out a wave of electric force, even though the electron itself, or rather its core, be at rest. Again, when the uniformity of distribution is destroyed by the passage of a cathode ray, after the ray has passed out the non-uniform distribution will not be in equilibrium, but will vibrate about the uniform one or else pass off as a wave into space. Thus an electron, and also a proton, when regarded as an assemblage of lines of force, has definite periods of

* Communicated by the Author.

vibration associated with it; these periods are natural constants, and, as we shall see, influence the wave-lengths of the waves associated with an electron. Thus, behind the electron and the proton there is a structure which can transmit and absorb electrical waves, even though the centre of it is at rest, and which has definite periods of vibration. Lines of force will thus modify the properties, the optical properties for example, of the medium through which they pass, so that electrons and protons are the centres of regions of exceptional optical properties.

In two papers, one on "The Scattering of Light by Electrons," Phil. Mag. xl. p. 713 (1920), and the other on "Mass, Energy, and Radiation," Phil. Mag. xxxix. p. 679, I investigated the effect produced by free electrons on electrical waves by calculating directly the secondary waves emitted by the electrons when set in motion by the electric force in the primary wave, and then combining these with the primary wave. It was shown that, when the primary wave first enters the medium, the front advances with the velocity c, that of light through a vacuum. Soon, however, the secondary waves coming from the electrons unite with the primary waves and produce a new distribution of electric force in the wave. In a short time things settle down into a new regime; and when the free periods of the electrons are long compared with the period of the light, the distance between two places in the wave which are in the same phase is greater than that in the incident wave before it struck the electrons. Thus the phase-velocity, which is defined to be the quotient of the distance between two places in the same phase by the time of vibration of light, is greater than c, though the wave system is the resultant of waves, all of which are propagated with the velocity c. The phase-velocity is a mathematical rather than a physical conception; the fact that it is greater than c does not involve that a sudden disturbance would be propagated through the medium at a speed greater than that of light.

It is shown in the papers referred to that this method leads to the same results as those deduced from Lorentz's theory of dispersion, so that the components of the electric and magnetic forces in the wave will satisfy equations of the form

$$c^2\left(\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}+\frac{d^2\psi}{dz^2}\right)=\frac{d^2\psi}{dt^2}+\Sigma\left\{\frac{p_r e^2/m\cdot\frac{d^2}{dt^2}}{\frac{d^2}{dt^2}+n_r^2}\right\}\psi,$$

where p_r is the number of electrons which have a frequency n_r.

When the frequencies of the waves are large compared with the free periods of the electrons, n_r^2 is small compared with d^2/dt^2, and the equation becomes

$$c^2\left(\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}+\frac{d^2\psi}{dz^2}\right)=\frac{d^2\psi}{dt^2}+(\Sigma(p_re^2/m))\psi.$$

When the density of the electrons is uniform, $\Sigma(p_re^2/m)$ is constant. Let it be denoted by B, then

$$c^2\left(\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}+\frac{d^2\psi}{dz^2}\right)=\frac{d^2\psi}{dt^2}+\mathrm{B}\psi. \quad . \quad . \quad (1)$$

If $\psi=\cos\frac{2\pi}{\lambda}(vt-x)$ is a solution,

$$v^2=c^2+\frac{\mathrm{B}}{4\pi^2}\lambda^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \quad . \quad (2)$$

where v is the phase-velocity of a wave of length λ.

We see from this equation that the radiation from free electrons endows a medium through which light is travelling with remarkable optical properties. These are :—

1. The phase-velocity is always greater than the velocity of light through a vacuum.
2. The phase-velocity increases with the wave-length and, when the waves are very long, is proportional to it.
3. A small change in the frequency corresponds to a large change in the wave-length.

When a medium possesses these properties we shall say that it is in a super-dispersive state.

We have supposed that the scattered secondary radiation which gives rise to this state was due to electrons, but it is evident that similar effects will be produced by anything which

(1) radiates electric force when moved ;
(2) is moved by the electric forces which exist in a wave of light.

The lines of electric forces which stretch from charged bodies possess these properties, and the regions round electrons and protons are crowded with such lines. Indeed, we regard an electron or a proton as an assemblage of lines of electric force, and as having a definite physical structure determined by the disposition of these lines, and not as a mere point charge without structure. Since the region round an electron or a proton is full of systems (the lines of

electric force) which can scatter the light, these regions are "super-dispersive" regions through which light is propagated according to very special laws. The disposition of these lines of force round the electron or proton will depend upon the law of force. If, as I have suggested, the law is more complicated than that of the inverse square, the structure of the field will be more favourable for the production of radiation than the simpler field.

We now proceed to consider the optical properties of these super-dispersive regions arising from the lines of force. The equations in the field are of the type of equation (1), where B is now a quantity depending on the density of the lines of force.

Since $$v^2=c^2+\frac{B}{4\pi^2}\lambda^2,$$

u the group-velocity, $v-\lambda\frac{dv}{d\lambda}$, is given by the equation

$$u=\frac{c^2}{\sqrt{c^2+\frac{B}{4\pi^2}\lambda^2}}=\frac{c^2}{v}; \quad \ldots \ldots \quad (3)$$

hence $$uv=c^2.$$

The wave-length corresponding to group-velocity u is by (3):

$$\sqrt{B}\,.\frac{\lambda}{2\pi}=c\sqrt{\frac{c^2}{u^2}-1}. \quad \ldots \ldots \quad (4)$$

The frequency n of waves of this length is given by

$$n=\frac{2\pi}{\lambda}v=\frac{\sqrt{B}\,.\,c}{\sqrt{c^2-u^2}}, \quad \ldots \ldots \quad (5)$$

and is thus nearly constant and equal to $\sqrt{B}$ unless u approaches c. $\sqrt{B}$ is the reciprocal of a time which is determined by the structure formed by the lines of force round the electron.

If μ is the refractive index of the medium,

$$\mu=\frac{c}{v}=\sqrt{1-\frac{B}{n^2}}. \quad \ldots \ldots \quad (6)$$

We can prove easily that no waves of smaller frequency than $\sqrt{B}$ can travel through the super-dispersive medium. For from equation (1) we have for plane waves

$$c^2\frac{d^2\psi}{dx^2}=\frac{d^2\psi}{dt^2}+B\psi;$$

if ψ varies as $\epsilon^{\iota pt}$,

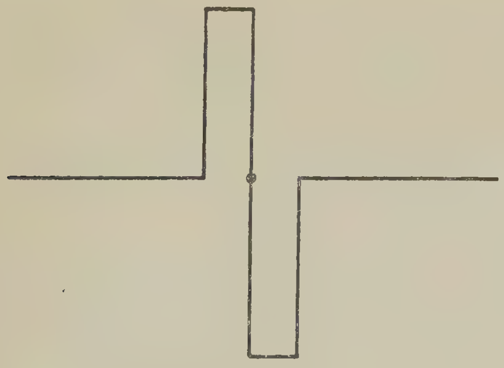
$$c^2 \frac{d^2\psi}{dx^2} = (\mathrm{B} - p^2)\psi;$$

hence, unless p^2 is greater than B, ψ will not be a harmonic function, and a wave will not pass through the medium.

The Electron in motion and the Waves which accompany it.

Let us consider from this standpoint the state of things round a moving electron. What is it that happens on this view when an electron is set in motion? We have to interpret the physical effects in terms of lines of electric force. The primary effect of an impulse is to alter the disposition of the lines of force. Before the impulse these were uniformly distributed round the electron; after it they are compressed in some places, dilated in others. Suppose, to fix our ideas, that the lines of force which before the impulse were uniformly distributed across a strip AB get transferred by the impulse from the right to the left of the strip, so that the curve

Fig. 1.



representing the disturbance in density would be represented by a graph like that in fig. 1. This non-uniform distribution is not a configuration of equilibrium, and, like a local strain in an elastic solid, travels out as a system of waves through the medium. To find the wave-lengths we must express by Fourier's series the initial disturbance as a system of harmonic terms, and these harmonics will travel out each with its appropriate velocity. If the wave-lengths involved are within narrow limits, there will be a series of waves of

O 2

phase of this wave-length starting from the electron, and thus the moving electron will be associated with a system of waves. These waves will show the same interference effects as Röntgen rays, and when they pass through a crystalline lattice will have their energy concentrated in definite directions, the energy associated with the electron following them in these directions. The wave-length of these will depend upon the distribution of the original disturbance; for example, in the case just considered it would be a quantity of the order of the breadth of the slit. When the Fourier waves are within small limits of wave-length, any disturbance such as that represented in the diagram will travel with the group-velocity corresponding to this wave-length. The disturbance consists of a special arrangement of the lines of electric force, and these lines travel along with it; and since the electron is just the termination of these lines, it travels along with them. Thus we conclude that the velocity of the electron is the group-velocity of a system of waves.

I may remark in passing that on the view expressed in my paper on "Mass Energy and Radiation" the kinetic energy of the electron is not due to any particular mass moving with the velocity of the electron, but because the lines of force, when arranged so as to produce a disturbance which travels with this velocity, grip more of the energy in the surrounding field than when they are uniformly distributed: the excess of energy is the kinetic energy of the electron.

To put the reasoning into a more analytical form, the disturbance at the time x and t can by Fourier's theorem be expressed in the form

$$\int_0^{\infty} \phi(k) \cos k(x-vt-\alpha)dk, \quad . \quad . \quad . \quad . \quad (7)$$

where $$\phi(k)=\int_{-\infty}^{+\infty} f(\omega)\cos(k\omega+\beta)d\omega,$$

where $f(x)$ represents the disturbance at x when $t=0$.

When, as in our case, v the phase-velocity depends upon the wave-length $(2\pi/k)$, v in equation (1) must be regarded as a function of k. When the initial disturbance is confined to a small distance, $\phi(k)$ will have a decided maximum for a value of k of the order of $2\pi/a$, where a is the length over which the initial disturbance is spread. Thus the wave-length of the electronic waves will be $2\pi/k_1$, where k_1 is the value which makes $\phi(k)$ a maximum, and the velocity of the

electron will be the group-velocity corresponding to this wave-length.

We see from equation (4) that

$$\lambda u = \frac{2\pi}{\sqrt{B}} c^2 \sqrt{1 - u^2/c^2}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or if m is the mass of the electron when its velocity is u, m_0 the mass when it is at rest.

$$m\lambda u = \frac{2\pi}{\sqrt{B}} m_0 c^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Thus $m\lambda u$ should be constant. This relation is accurately satisfied in Professor G. P. Thomson's experiments, from which we can deduce the value of B. He finds that when $u = 10^{10}$, $\lambda = 7 \cdot 8 \times 10^{-10}$. Substituting these values in equation (8), we find

$$\frac{2\pi}{\sqrt{B}} = 9 \times 10^{-21}.$$

$2\pi/\sqrt{B}$ is a time depending on the structure of the lines of force.

We see from equation (5) that $\sqrt{B}$ is the frequency of the waves accompanying electrons whose velocity is small compared with c. Since $\sqrt{B} = 7 \times 10^{20}$, this frequency is comparable with that of the hardest γ-rays.

Equation (8) would be identical with the one given by de Broglie's theory if

$$\frac{2\pi}{\sqrt{B}} m_0 c^2 = h,$$

where h is Planck's constant.

If the lines of force in the electric field round an electron are discrete entities, so that the field possesses a finite structure both in time and space, we can see that the velocity of an electron under an electric field might increase by finite increments and not continuously. Thus, for example, if the electron had a finite structure in space, the breadth of such a disturbance as that represented in the figure might not be able to take all values, but only those of which an integral number were in a length fixed by the structure. In this case $1/\lambda$ would change discontinuously; and it follows from equation (9) that u, the velocity of the electron, would do the same. Thus, if the electric field associated

with an electron had a structure in time and space, we should expect to find results analogous to those attributed to quanta.

Waves near Moving Protons.

Since the proton is also the origin of a system of lines of force, we should expect that the motion of protons as well as of electrons should be accompanied by a train of waves, though, owing to the differences between the masses and dimensions of the proton and electron, the lengths of the waves and the frequencies for the same velocity would be different in the two cases. We should expect that B would be greater for the proton than for the electron, so that the frequency of the waves accompanying the proton would be greater than that of the electronic waves. On the assumption that $mu\lambda$ is constant, $\sqrt{B}$ is proportional to the mass of the moving charge, so that the frequency of the waves accompanying a moving proton would be about $1\cdot2\times10^{24}$.

Since B will depend on the density of the lines of force, it will be a function of the distance from the charged body. Thus the equations in the field round the charged body will be of the form

$$c^2\left(\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}+\frac{d^2\psi}{dz^2}\right)=\frac{d^2\psi}{dt^2}+f(r)\psi,$$

where r is the distance of the point xyz from the charged body. As the distance from the charged body increases, the value of B diminishes, so that by equation (5) the refractive index of the region round the charge will increase with r.

Thus a ray of light, if it is able to pass through this region, will be passing from an optically dense to an optically rare medium, and so will appear to be repelled by the charge.

In this paper the electron has been supposed to be moving uniformly. I hope on another occasion to discuss from the same point of view the motion of an electron under electric force.

XVIII. *The Calculation of the Equivalent Conductivity of Strong Electrolytes at Infinite Dilution.*—Part II. (i.) *Methyl Alcoholic Solutions.* (ii.) *The Effect of Temperature on the Constants in the Equation* $\Lambda_0 = \Lambda + BC^n$.

By ISRAEL VOGEL, *M.Sc., D.I.C., Beit Scientific Research Fellow* *.

IN previous communications (Ferguson and Vogel, this Journal, 1925, l. p. 971; 1927, iv. pp. 1, 233, 300; compare also Trans. Far. Soc. 1927, xxiii. p. 404) the formula

$$\Lambda_0 = \Lambda + BC^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

was applied to the calculation of the equivalent conductivity at infinite dilution, Λ_0, of aqueous solutions of strong electrolytes of very varied types. The results obtained may be summarised as follows:—

(*a*) The constants B and n vary from electrolyte to electrolyte and in a regular manner for related electrolytes.

(*b*) The extreme variations n lie between 0·3742 for potassium chloride at 25° and 0·9687 for iodic acid at 25°.

(*c*) The constants B and n vary with temperature, although owing to the scarcity of reliable data no definite conclusions could be drawn as to the exact mode of variation.

(*d*) The formula is applicable to uni-uni, uni-bi, and bi-bivalent electrolytes, and also to such strong acids as hydrochloric and iodic acids.

These results, which can be taken as an expression of the facts of observation, are not in agreement with the requirements of the complete ionization theory of Debye and Hückel (*Phys. Zeit.* 1923, xxiv. p. 305; compare also Debye, Trans. Far. Soc. 1927, xxiii. p. 334; Onsager, *ibid.* 1927, xxiii. p. 341), which leads to an equation of the form

$$\Lambda_0 = \Lambda + PC^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

i.e. the exponent of C, which is equal to $\frac{1}{2}$, is independent of the nature of the electrolyte, of temperature, and of the solvent. This equation has been extensively employed for the extrapolation of Λ_0 in non-aqueous solutions (compare

* Communicated by the Author

Walden, *Z. anorg. Chem.* 1920, cxv. p. 49). Formulæ in which the exponent of C is other than $\frac{1}{2}$ have also been employed, such as $\frac{1}{3}$ (Philip and Courtman, J. Chem. Soc. 1910, xcvii. p. 126 ; King and Partington, Trans. Far. Soc. 1927, xxiii. pp. 522, 531), 0·45 (Lorenz, *Z. anorg. Chem.* 1919, cviii. p. 191 ; Walden, *ibid.* 1920, cxv. p. 49) and unity—Ostwald dilution law—(Kraus and Bishop, J. Amer. Chem. Soc. 1922, xliv. p. 2206 ; Morgan and Lammert, *ibid.* 1924, xlvi. p. 1117). These are all, however, special cases of the general formula (1), and all involve an *a priori* assumption as to the value and constancy of the exponent n. It appeared of interest, therefore, to extend the calculations to non-aqueous solutions, and the present paper is concerned with (i.) solutions in methyl alcohol (a solvent closely related to water), and (ii.) the effect of temperature.

Methyl Alcoholic Solutions.

The data employed were those of Frazer and Hartley at 25° (Proc. Roy. Soc. A, 1925, cix. p. 351), and Λ_0 was calculated by the method previously described (this Journal, 1925, l. p. 971), the values of Λ at the requisite concentrations being *interpolated* from a large-scale Λ–C graph In view of the small range of concentration available for the calculation of Λ_0—it was usually between 0·0002 to 0·002 normal,—the value of r used was 1·5, and in this way values for B and n were obtained more reliable than those given in preliminary computations (Trans. Far. Soc. 1927, xxiii. p. 414), in which r was taken as 2. The actual results are given in full in Table I. below ; those for lithium nitrate are somewhat doubtful, owing to a slight ambiguity in the exact position of the conductivity-concentration curve. Potassium nitrate has been omitted, since the results for both runs do not lie on one Λ–C curve.

Table I.

Concentration, $\times 10^{-4}$.	LiCl. Λ.	LiCl. Λ_0.	NaCl. Λ.	NaCl. Λ_0.	KCl. Λ.	KCl. Λ_0.
2·000	87·66	89·91	93·64	97·24	101·35	104·95
3·000	87·02	89·92	92·94	97·29	100·55	104·97
4·500	86·19	89·92	92·03	97·29	99·54	104·97
6·750	85·11	89·91	90.94	97·30	98·27	104·94
10·13	83·75	89·92	89·63	97·31	96·75	104·95
15·20	82·15	90·09	88·02	97·31	94·89	104·96

TABLE I. (*continued*).

Concentration, $\times 10^{-4}$.	RbCl. Λ.	RbCl. Λ_0.	CsCl. Λ.	CsCl. Λ_0.	$LiNo_3$. Λ.	$LiNo_3$. Λ_0.
2·000	104·69	109·42	109·43	113·73	96·59	100·22
3·000	103·76	109·42	108·54	113·79	95·79	100·22
4·500	102·65	109·43	107·39	113·78	94·82	100·23
6·750	101·26	109·39	105·99	113·78	93·63	100·23
10·13	99·65	109·38	104·30	113·80	92·16	100·22
15·20	97·74	109·39	102·20	113·78	90·53	100·37

Concentration, $\times 10^{-4}$.	$NaNO_3$. Λ.	$NaNO_3$. Λ_0.	$RbNO_3$. Λ.	$RbNO_3$. Λ_0.	$CsNO_3$. Λ.	$CsNO_3$. Λ_0.
2·000	102·36	105·99	113·00	117·19	117·65	123·25
3·000	101·47	105·99	111·93	117·19	116·39	123·23
4·500	100·36	105·99	110·59	117·19	114·88	123·24
6·750	99·00	106·01	108·90	117·19		
10·13	97·24	105·96	106·82	117·22		
15·20	95·20	106·06				

Concentration, $\times 10^{-4}$.	$AgNO_3$. Λ.	$AgNO_3$. Λ_0.	KBr. Λ.	KBr. Λ_0.	KI. Λ.	KI. Λ_0.
2·000	106·08	109·97	105·69	109·17	111·11	114·38
3·000	104·89	109·97	104·86	109·19	110·31	114·39
4·500	103·33	109·97	103·82	109·19	109·29	114·37
6·750	101·30	109 97	102·57	109·17	108·04	114·37
10·13	98·65	109·97	101·01	109·19	106·50	114·38
15·20	95·40	(110·20)	99·18	109·29		

It will be seen that the consistency of Λ_0 leaves little to be desired. A summary of the results is shown in Table II.;

TABLE II.

Salt.	Λ_0 (Author).	B.	n.	Λ_0 (Frazer & Hartley).
LiCl	89·92	447·6	0·6213	90·90
NaCl	97·29	192·3	0·4670	90·95
KCl	104·96	271·2	0·5074	105·05
RbCl	109·41	209·0	0·4448	108·65
CsCl	113·78	275·2	0·4882	113·60
$LiNO_3$	100·22	240·7	0·4926	100·25
$NaNO_3$	106·00	363·5	0·5409	106·45
$RbNO_3$	117·20	495·8	0·5604	118·15
$CsNO_3$	123·24	373·3	0·4930	122·95
$AgNO_3$	109·97	1066·1	0·6591	112·95
KBr	109·20	300·3	0·5226	109·35
KI	114·38	331·4	0·5422	114·85

for purposes of comparison the values of Λ_0 calculated by Frazer and Hartley (*loc. cit.*) by means of the square-root formula are included.

The superiority of the general equation (1) over the widely-used square-root formula (2) is evident from Table III. below, in which are tabulated the values of Λ_0 for silver nitrate over a range of concentration.

Table III.—$AgNO_3$ 25° C.

Concentration, $\times 10^{-4}$.	$\Lambda_0 = \Lambda + 1066{\cdot}1C^{0{\cdot}6591}$.	$\Lambda = \Lambda + 451C^{0{\cdot}5000}$.
2·000............	109·97	112·46
3·000............	109·97	112·70
4·500............	109·97	112·90
6·750............	109·97	113·02
10·13	109·97	113·00
15·20	110·20	112·98

The results of a test of Kohlrausch's law of independent migration of ions are given in Table IV. No explanation is at present offered of the irregularities, although it should be stated that Frazer and Hartley obtained good agreement by using values of Λ_0 computed from the formula (2)*.

Table IV.

Cation.	Li.	Na.	Rb.	Cs.
Λ_0 nitrate $-$ Λ_0 chloride	10·30	8·71	7·79	9·46

The Effect of Temperature on the Constants B *and* n *in the Equation* $\Lambda_0 = \Lambda + BC^n$.

Practically the only reliable data for the conductivities of salts in non-aqueous solutions over a wide range of temperature appear to be those of Philip and Oakley (J. Chem. Soc. 1924, cxxv. p. 1189; compare, however, McBain and Coleman, Trans. Far. Soc. 1919, xv. p. 27; Robertson and Acree, J. Phys. Chem. 1915, xix. p. 413) for potassium iodide in nitromethane. Their results are expressed in the form of a table in which are tabulated the values of the dilution in litres V, the specific conductivity κ. and the equivalent conductivity Λ (Philip and Oakley, *loc. cit.*

* Compare, however, Part I. (Phil. Mag. 1925, l. p. 971) for aqueous solutions, in which by application of formula (1) better agreement for Kohlrausch's law was obtained than that deduced by Kohlrausch and Maltby from equation (2).

table 1). The values of Λ employed by the author were those obtained by multiplying the dilution in litres V by the specific conductivity κ, except for the figures at 25°, where the tabulated values of Λ were used, since the product $V\kappa=\Lambda$ leads to obviously untenable values of the equivalent conductivity. The complete results are presented in Table V., while a conspectus is given in Table VI. below.

Table V.—KI in Nitrometheane.

Concentration, $\times 10^{-4}$.	0°. Λ.	0°. Λ_0.	25°. Λ.	25°. Λ_0.	40°. Λ.	40°. Λ_0.
2·000	88·10	90·80	119·30	123·01	139·5	144·1
3·000	87·41	90·81	118·27	123·02	138·3	144·1
4·500	86·52	90·81	116·94	123·01	136·7	144·1
6·750	85·35	90·76	115·27	123·04	134·7	144·1
10·13	83·94	90·77	113·10	123·05	132·1	144·1
15·20	82·15	90·76	110·27	123·00	128·8	144·1

Concentration, $\times 10^{-4}$.	55°. Λ.	55°. Λ_0.	70°. Λ.	70°. Λ_0.
2·000	160·53	165·30	184·8	193·0
3·000	159·17	165·33	182·7	193·0
4·500	157·36	165·31	180·1	193·0
6·750	154·99	165·27	176·8	192·9
10·13	151·98	165·26	172·8	193·0
15·20	148·01	165·16	167·8	193·1

Table VI.—KI in Nitromethane.

Temperature.	Λ_0.	B.	n.
0°	90·79	353·8	0·5726
25°	123·02	659·6	0·6083
40°	144·1	717·6	0·5939
55°	165·27	1034·5	0·6317
70°	193·0	928·6	0·5552

Both the constants B and n increase steadily with temperature up to 55° *, and then pass through a maximum between this temperature and 70°. Whether this manner of variation is general in non-aqueous solutions or is peculiar to nitromethane is a question which can only be answered when more experimental data for the conductivities of salts in non-aqueous solutions over a wide range of temperature are available.

* Except for the value of n at 25°.

On plotting the values of Λ_0 here calculated against the reciprocals of the viscosities or fluidities determined by Philip and Oakley (*loc. cit.*), a straight line is obtained, indicating that the relationship between the conductivity at infinite dilution and the fluidity f over the range of temperature 0–70° is of the form

$$\Lambda_0 = a + bf,$$

where a and b are constants.

In conclusion, the author wishes to thank the Trustees of the Beit Scientific Research Fellowships for a Fellowship, Dr. A. Ferguson for his kind interest, and the Dixon Fund of the University of London for a grant which has defrayed most of the expenses of the research.

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XIX. *Interference between Grating-Ghosts.* *By* Sven Fagerberg, *Upsala* *.

[Plate IV.]

It is well known that the illuminated field behind the focus of a grating spectrograph generally presents a characteristic appearance as to the distribution of intensity. Surveying the illuminated field behind the focus with an eyepiece at a sufficient distance from the focus, one finds the field traversed by several closely-lying sharp luminous lines separated by dark intervals parallel to the spectral lines. The arrangement of the lines is rather irregular.

In order to investigate this fact more closely, the wavefront of a single spectral line (Fe $\lambda = 4384$ T. A, third order) was photographed. The line was separated from the remaining part of the spectrum by a slit in the focal plane, and a photographic plate was placed in the way of the diverging beam.

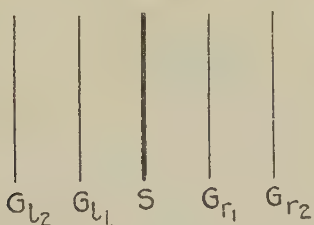
The spectrograph used consists of a Rowland plane grating (AE″, 14,438 lines to the inch, ruled in the year 1889), rotating about an axis perpendicular to the collimator and camera. These are fixed at an angle of about 35° to each other. The focal distance of the lenses is 165 cm., and the plate was placed about 95 cm. behind the focal plane.

* Communicated by Prof. Manne Siegbahn, D.Sc.

If only one spectral line passes the slit in the focal plane, the wave-front has the appearance shown in A (Pl. IV.). From this it will be seen that the variations of intensity along the wave-front are quite irregular and not very definitely marked.

The grating in question shows, however, in the third order, in addition to the principal line of high intensity, fainter lines, the so-called ghosts, which are of the same wave-length as the principal line and are symmetrically located about it. The arrangement of the lines appears from fig. 1. S is the principal spectral line, G_{r_1}, G_{r_2} and G_{l_1}, G_{l_2} the ghosts of the first and second orders to the right and left. The ghosts and the principal line are all equidistant.

Fig. 1.



Now widening the slit in the focal plane in order to transmit also the ghost of the first order on one side, the wave-front shows a totally different appearance (C, Pl. IV.). One obtains a number of equidistant luminous lines separated by dark intervals. On the other hand, the ghost alone gives a wave-front, in its general character agreeing with that of the principal line (B, Pl. IV.).

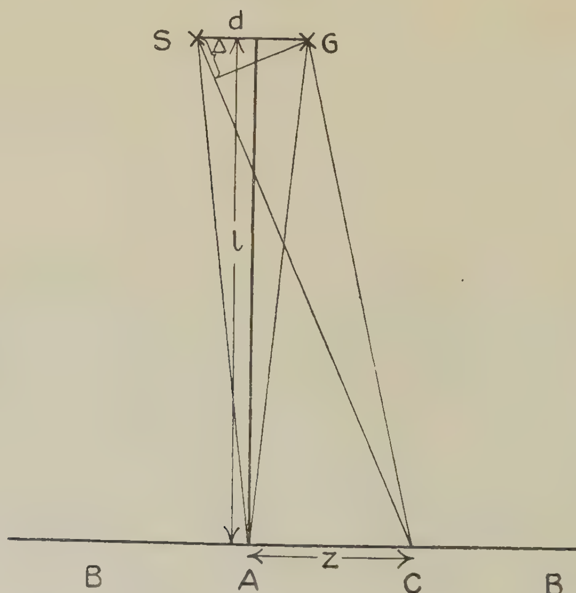
It is reasonable to suppose that the appearance of the luminous and dark lines in the case when the principal line and the ghost both pass the slit is due to interference behind the two diverging beams when they are superposed between the focus. Thus we may regard the spectral line and its ghost as two linear coherent sources of light.

The elementary treatment of the phenomenon (see fig. 2) is entirely analogous to the treatment of, for example, the Fresnel mirrors. In the diagram the points S and G are the projections of the spectral line and the ghost on a plane perpendicular to them, and AB indicates the photographic plate. In A the rays have travelled the equal path, and therefore reinforce each other, provided that

they are in the same phase in S and G. In C the beams have the path-difference $\Delta = \frac{dz}{l}$ (see fig. 2), supposing that d and z are small in comparison with l, this also in reality being the case.

Thus the beams will reinforce each other when $\Delta = \frac{dz}{l} = n\lambda$, λ being the wave-length and $n = 1, 2, 3 \ldots$ Consequently, the position of a luminous line is $z = n\frac{\lambda l}{d}$. On the plate we get equidistant lines at the distance of $a = \frac{\lambda l}{d}$ from each other.

Fig. 2.



In order to test the correctness of the above assumption, the distances a, l, and d were measured. The position of the focal plane was determined, successively photographing the spectrum with different adjustments of the camera slide, and in each case noting the reading on the camera scale. Examining these photographs, the one which showed the spectral lines with the greatest sharpness was chosen. The corresponding reading on the camera scale gave the position

of the focal plane. On this plate the distance between the principal line and ghost was measured.

In order to obtain a good value of a, the distance between every tenth interference fringe was measured. It appears, however, that the distance between two adjacent fringes is subject to small accidental variations with their position on the plate. The mean distance determined in the way described above is, on the contrary, constant within a wide region in the middle of the plate.

The following values were obtained :—

$$a = 0{\cdot}554 \text{ mm.}$$
$$d = 0{\cdot}750 \text{ ,,}$$
$$l = 943{\cdot}7 \text{ ,,}$$

Hence $\lambda = 440 \pm 4\ \mu\mu.$

Standard value $\lambda_s = 438{\cdot}4\ \mu\mu.$

The agreement is undoubtedly good.

In order to get another control, the slit in the focal plane was widened to transmit also the ghost of the second order on the same side of the principal line. As the distances between the principal line and the ghosts are equal, the appearance of the interference phenomenon (provided the three rays are in the same phase when passing the focus) must be the same as in the preceding case, with the distinction that fainter maxima of intensity are to occur between those already observed. This is also the case, as seen from D (Pl. IV.).

Making the principal line and *all* the ghosts to interfere, more complicated circumstances take place (E, Pl. IV.). As already stated, the grating in question gives (in the third order) ghosts of the first and second order on both sides of the principal line. Thus, if the whole complex passes the slit, five beams are brought to interference. On account of the rather irregular structure of this plate, it seems as if all waves do not pass the focus in the same phase, and the wave-fronts from the grating are not completely plane, they being subject to some small divergencies due to errors in the spacing of the grating. These plates, where more than two rays have been brought to interference, have much in common with photographs carried out by Dowling and Teegan (Phil. Mag. l. p. 241, 1925). In order to obtain several coherent sources, they used a complex of bi-prisms, which thus gave birth to interference fringes of the same kind as those described above.

At first it seems surprising that the plates can show the interference so markedly, using beams of intensities so different as a spectral line and its ghosts. A rough estimate of the ratio between the intensity of the spectral line and that of the ghost of the first order gives, however, 25 : 1. The ratio between the amplitudes is thus 5 : 1, and the ratio between maximum and minimum intensity in the case of interference $(6:4)^2=9:4$.

A phenomenon analogous to that above described has been observed by Wood (Phil. Mag. xlviii. p. 497, 1924), who worked with a small concave grating. He illuminated the collimator slit by monochromatic light, and placed the plate *between* the grating and the focus. Assuming that the circumstances are symmetrical with respect to focus, the lines observed by him would correspond to the interference fringes treated above, evidently with the difference that Wood was not able with his arrangement to shut out the ghosts, but observed the resulting effect of the spectral line and all its ghosts. Thus he did not photograph the wave-front of the spectral line alone (see A, Pl. IV.). He however, is of the opinion that the observed luminous lines belong to this wave-front. According to this conception, the periodical errors of the grating have influenced the wave front of the principal line in such a way as if it had passed a system of parallel slits in an opaque screen, and the ghosts are regarded as spectra of different orders of this assumed system of slits.

Upsala, Physical Laboratory
of the University,
Oct. 1927.

XX. *Prof. Joly and the Earth's Thermal History.* *By* HAROLD JEFFREYS, *F.R.S.**

PROF. JOLY uses an extract from a paper of mine in the 'Geological Magazine' for November 1926 as the basis for a further discussion†. He would have done better to choose a different passage, as follows (p. 521) :—

"In any case the onus of proof that the theory of magnetic cycles explains anything at all rests with its supporters, at least if the word 'explain' is understood in the sense given to it by physicists. A theory explains a fact if it starts from

* Communicated by the Author.
† Phil. Mag. August 1927.

a set of stated hypotheses and shows how the fact in question follows from them. But the theory of magnetic cycles considers itself at liberty to use the observed facts to enable it to infer its own consequences; and any fact can be explained on any theory if that is permitted. If resolidification under excess of internal heat is possible, it should be well within the experimental powers of both Professor Joly and Professor Holmes to demonstrate it; without this no case for the theory has been made out."

From Prof. Joly's omission to mention this passage many readers may be inclined to draw their own conclusions. But I think that further remarks on elaborate attempts such as his to solve difficult problems in theoretical physics with no mathematics beyond simple proportion are relevant at this stage, especially since he accuses me of misrepresenting his views. My answer to this charge is that I have devoted a great deal of time to trying to understand his theory, that I am as well qualified to understand it as most people, and that if I have misinterpreted it in any respect the reason is that it is unintelligible, and that the reason for its unintelligibility is his omission to state in proper form the laws he assumes.

A specimen of the difficulty of seeing what Prof. Joly really means is provided by his assertion (p. 338) that "the heat is latent, and there is but little variation of temperature," maintaining at the same time that his object was not what I said, namely to explain a periodic variation of temperature. Yet he quotes, apparently with approval, on p. 340, a passage from Mr. J. R. Cotter, as follows: "Prof. Joly thinks that the θ_0 isotherm (that where the temperature is equal to the melting-point) would rise to within a very few miles of the surface. It would, however, finally become stationary, and then descend. . . . The whole series of changes would form a cycle which would be repeated." If this is not a periodic variation of temperature I do not know what is. Again, in 'The Surface History of the Earth,' p. 105, we have: "In periods of liquefaction and tidal movements the floor must be at first rapidly reduced in thickness. For, in fact, much of it is near the melting-point. . . . It seems certain that the ocean floor cannot be indefinitely reduced in thickness. There will be a certain limiting thickness which will be attained when the rate of escape of heat into the ocean becomes equal to the rate of supply from beneath." When the abnormally large supply of heat is being conducted through the abnormally thin ocean floor an abnormally high temperature gradient

must exist. This is one of the things I meant in referring to variation of underground temperature.

Anyhow, my objection to periodicity in the flux of heat is equally strong whether or not it is associated with periodicity in temperature.

I gave in a previous paper * two diagrams indicating the course of events to be expected if there is an excess of heating underground. Two were necessary on account of ambiguities as to the initial state assumed by Prof. Joly. He has not pointed out where, if anywhere, he disagrees with these diagrams, nor has he offered any alternative. He does try to dismiss my analogy with the motion of a stretched string subject to damping as misleading, but does not say explicitly what alterations are to be made in the inferences.

He remarks about this analogy † that there is nothing in it analogous to the accumulation of energy in the form of latent heat or to the transfer of this energy by convection. Both phenomena are fully allowed for in the analogy. In the thermal problem the fluid region is represented by the portion of the string in contact with the rigid barrier. The rate of extension of this region is given by the condition that the amount of fluid melting, multiplied by the latent heat of fusion, is equal to the excess of the heat generated within the fluid over that conducted out at the top and bottom. Since convection transfers heat upwards, the growth downwards will not extend below the bottom of the radioactive region ; the partition between growth at the bottom and the top before that stage is reached could be stated in a given case. The string allows for these phenomena automatically if, as I explicitly stated, the transverse force originally applied to the constrained region is transferred to its upper margin. Convection again is allowed for by the supposition that the distribution of temperature in the fused region is the adiabatic for a mixture of solid and liquid ; this is expressed by the slope of the barrier.

Specimens of Prof. Joly's way of dealing with opposition are to be found on pp. 934–5 of his paper of May 1926. He says that the differential equation of heat conduction used by me is the same as that used by Mr. Cotter in 1924, save that I assume the rate of supply of heat constant. "Of course the time should enter through a negative exponential factor." Is it seriously suggested that this equation was not published till 1924, or that the decay of uranium and thorium during

* Phil. Mag. i. pp. 923–931 (1926).
† Phil. Mag. i. p. 934 (1926).

geological time is of any importance in the present problem? Mr. Cotter himself takes the rate constant.

My doubts as to the reason why Prof. Joly assumes fusion to occur under the oceans, as expressed in the first two paragraphs on p. 927 of my paper, are treated by him on p. 935. Direct comparison of the two passages is the only way of judging how far he has answered. I would add only that a flat contradiction on a question of fact, without a reference, is not in accordance with the best scientific practice. The views I expressed agree with those of J. H. L. Vogt *.

Prof. Joly's next paragraph contains the elegant remark: "We have here the mathematical point of view with a vengeance." It does not, however, answer my statement that "radioactivity within the granitic layer . . . only ensures that the temperatures below that layer become higher than they would in its absence." Incidentally, this will produce the further effect of diminishing viscosity and enabling convection to bring up the heat generated below with greater ease and at an earlier stage, so that its escape is made easier, not obstructed.

His paper opens with a reply to my arguments that the thickness of the granitic layer does not exceed about 15 km.; but as all his arguments are now out of date and mine have received additional support †, I do not propose to deal with them further.

Prof. Joly's new paper (Aug. 1927) offers a series of periodic phenomena due to a steady supply of heat, and considers that they provide an answer to my statement that periodic phenomena cannot be produced in this way. Perhaps I ought to have added a list of exceptional cases where the proposition might not be expected to hold, as is done at great length in works on pure mathematics. But in physics the discovery of these is usually left to the reader. After all, the whole science of engineering is devoted to constructing systems that will, if suitably treated, behave in an extraordinary way. But in the present problem we are entitled to suppose that the system has not been so designed. Prof. Joly has indeed done a service in mentioning the four examples he does, because they are admirable examples of the kind of exception that proves the rule. But as the conditions of convection in a layer of fluid of great horizontal extent are by this time fairly well understood, and as

* Econ. Geol. 1926, pp. 207–233. *Cf.* also 'Nature,' March 5, 1927, p. 343.

† M. N. R. A. S. Geoph. Suppl. i. pp. 385–402, 483–494. Gerlands Beiträge z. Geophysik. xviii. pp. 1–29 (1927).

the basaltic layer, if fused, would be such a layer, we may as well consider the actual case first. It is, of course, well known that if a fluid is incompressible, but the density increases with height on account of variation of temperature or composition, the potential energy can be decreased by interchanging upper and lower layers. This is not, however, a sufficient condition for instability, though it was long believed to be one. Direct interchange of the layers is impossible because they obstruct each other; it can occur only by upward currents in some places combined with downward ones in others, and these are hindered by conduction and viscosity. Consequently instability does not actually arise until the vertical gradient of temperature exceeds a certain finite value. When compressibility is taken into account, instability arises when the gradient exceeds the adiabatic by a certain amount. When the system first becomes unstable, the motion takes the form of a set of polygonal cells, fluid rising in the centres and sinking around the edges. The motion is continuous and steady. This was found experimentally by Bénard, and contributions to the theory have been made by Lord Rayleigh and others*. With increasing gradient of temperature other types of instability arise and other modes of motion become superposed, till we get the complete turbulence known in the ordinary heating of a liquid from below.

Now the horizontal extent of the cells is comparable with the depth. Also the tendency of the currents is to restore the adiabatic gradient of temperature; and the departure from it needed to start them is proportional to the inverse fourth power of the depth, and is negligible when the depth is of the order of a few kilometres. Consequently we must infer that when a fluid is heated at the bottom or internally, the convection currents generated will prevent the gradient of temperature from ever departing far from the adiabatic. Their effect is to carry the heat to the top as fast as it is supplied; increasing the rate of supply increases the intensity of the convective circulation, but not the gradient of temperature. But here we come to the essential point: the heat, on account of the limited size of the cells, always reaches the top within a horizontal distance of the place where it was originally supplied comparable with the depth of the liquid. At the top the liquid is in contact, in the geophysical problem, with the solid upper layer, the bottom surface of which must be at the same temperature as the top of the liquid. In the

* *Cf.* Phil. Mag. ii. pp. 833–844 (1926).

upper layer the heat is transmitted to the outer surface by conduction. The whole thickness of the two layers being of the order of 30 km., a region whose horizontal extent is over, say, 100 km. will be thermally practically self-contained. Mutual influence of continental and oceanic conditions will be negligible.

In the terrestrial case the transference of heat will be partly by way of latent heat; some liquid will solidify when it rises to the top, and the descending currents will contain suspended solid particles, which will gradually melt as they descend. But this does not affect the argument; the only difference is that in the region where solid is present the gradient is the adiabatic for a mixture of solid and liquid, and not that for a liquid alone. There is no possibility of what Prof. Joly calls superheating—that is, rise of temperature far above the melting-point *. These inferences are at all points in agreement with those I made previously and illustrated by diagrams. Convection simply straightens out part of the graph of temperature against depth, and in no respect facilitates periodicity.

Of the four self-acting periodic systems considered by Professor Joly, three differ from the geophysical problem through the working fluid being enclosed in a tube. I refer to the geyser, the Air Tester, and E. H. Griffiths's heat engine. The essence of the geophysical problem is the possibility of interchange between layers of the liquid by ascending currents in some places and descending ones elsewhere. In a tube this is obstructed, and far greater gradients of temperature are needed to start convection. On account of the constricted character of the systems, the liquid in two of them acts as a movable valve capable of opening and closing communication between two masses of gas; but in the geophysical one communication is open all the time. Griffiths's engine, with a trifling change in the constants, is converted into the ordinary apparatus for finding the coefficient of increase of pressure at constant volume. In Professor Joly's experiment with the rotating sphere containing camphor and heated below, we have again a system with only a single infinity of possible positions of equilibrium, of which only two are realized in practice, so that the system can only pass from one to the other discontinuously. The greater freedom of an extended horizontal

* In Preston's 'Heat,' 1919, pp. 344–348, this term is used for a rise in temperature above the normal boiling-point. What Professor Joly calls a superheated liquid is, in fact, an ordinary liquid.

layer makes continuous interchange possible; and this is what actually occurs. A better analogy with the conditions within the earth's crust would indeed have been the boiling of the domestic kettle or, still better, the melting of a tray of paraffin wax by heating below.

At the close of my former paper in this Magazine I stated four objections to Professor Joly's theory, any one of which would by itself be decisive. He has entirely refused to face three of them, and partly the other (*c*). But I would point out that my main objection to the theory is not merely the existence of serious arguments against it; it is that there are no serious arguments for it. The question is essentially one of theoretical physics: given a certain set of assumptions, to investigate their consequences in the light of physical laws and to compare the results with observation. This investigation will necessarily involve a considerable amount of mathematics; at the least the equations of fluid motion and of heat conduction must appear in it and be solved. But until it is carried out, there is no reason to believe that the consequences would be anything like those asserted by Professor Joly. The onus of proof in a question of theoretical physics is on the advocate of a theory, not on its opponents; and it ought in this case to have been carried out before Professor Joly proceeded to publication in an important physical journal and followed it up with an ambitious book. If the problems to be solved are in any way more complex than I have considered, so much the more complex must the problem be that Professor Joly has to solve before his theory will command the respect of physicists. The reason for this situation is simple: inadequately tested theories can be constructed more rapidly than people in the habit of taking reasonable care can point out the mistakes. I am sorry to have to point this out, but the number of suggestions that have been made in public and in private by non-physicists, to the effect that it is the duty of a mathematical physicist to drop any work he may consider more important in order to examine some theory of the type under consideration—with the certainty that the results will be accepted only if they are favourable—have made it necessary. I add that, since I have already devoted more attention to Professor Joly's theory than I consider it worth, I do not propose to take any part in further discussion of it unless he produces more satisfactory arguments than he has done hitherto.

XXI. *The Earth's Thermal History.*

By J. Joly, *F.R.S.**

MY paper entitled "Dr. Jeffreys and the Earth's Thermal History" (Phil. Mag. August 1927) has stirred Dr. Jeffreys into a rejoinder revealing so much irritation that I feel a real difficulty in pursuing the matter any further lest quite unintentionally I should say anything which might prove to be a source of further annoyance. At the same time he has raised issues which, not only for the advance of the subject under discussion, but as matter of courtesy, seem to render a further statement from me imperative.

My paper was based upon one appearing from the pen of Dr. Jeffreys in the 'Geological Magazine' of November 1926, in which he enunciates his "main conclusion." I cite this statement in my paper and proceed to comment upon it, pointing out that it appeared tacitly to assume that no relative motion of the parts of the medium takes place. Mr. Cotter's clear statement of what is involved in the theory of thermal cycles follows; and then certain phenomena are cited as showing that cyclic thermal phenomena might take place without involving any departure from the "main conclusion" arrived at by Dr. Jeffreys.

To this procedure Dr. Jeffreys takes exception in his paper appearing in this issue of the Philosophical Magazine. His first complaint is that I make the formal statement of his main conclusion the subject of my remarks, instead of some general observations of his, which appear later, as to the nature of "a theory" and how its supporters should defend it.

I think most scientific readers would regard it as more incumbent upon me—and probably more fruitful—to deal with the "main conclusion" than to embark upon the very wide topic favoured by Dr. Jeffreys.

Dr. Jeffreys objects to the analogy of the geyser. He says in effect—"Without boundary constraints there would be no cyclic effects in the case of the geyser, and therefore it 'proves the rule'—*i. e.* that such effects cannot arise."

It is true that there will be no cyclic effects without constraints. But the statement overlooks the fact that the cyclic nature of the phenomena turns on constraints, not only in the case of the geyser, but also in the case of the substratum; and of necessity must do so in all similar cases where storage of heat followed by change of state occurs.

* Communicated by the Author.

In the case of the geyser the medium is fluid, and constraints hindering free circulation are required while the energy necessary to attain the boiling-point at all levels is accumulating. In the case of the substratum the medium is solid to start with, and the corresponding constraint is in the rigidity of the working substance which determines quiescence while the energy necessary to attain the melting-point at all levels is accumulating. The difference in the character of the constraints, which arises necessarily out of the differing mobility of the media, possesses no fundamental significance. As thermal energy accumulates, similar conditions of instability arise in both cases—*i. e.*, a medium charged with energy proper to the pressure at all levels and necessarily changing state if any movement occurs to bring any part of it to a higher level.

The use of the word "rigidity" in connexion with the substratum should be qualified. For, in fact, every stage of viscosity must, in the substratum, successively prevail during long periods of time. Siliceous rocks have no definite melting-point. As thermal storage progresses, their viscous resistance to movement diminishes, the heat becoming latent in the gradual change of state. These conditions exist at all levels, but in the depths the thermal storage is greatest as higher melting-points have to be attained before movement can occur. Equilibrium finally breaks down, either spontaneously or due to tidal forces. The ensuing circulation involves the superheating and perfect fluidity of rising magma brought into regions of lower pressure. The possibility of storage turns, throughout, upon the resistance to translatory movement during a certain time-interval. The same may be said of the geyser, where the necessary constraints are not involved in the properties of the substance, but are, as it were, "external."

In short, the geyser, or its experimental realization, is a fair and direct demonstration of the inapplicability of Dr. Jeffreys's "main conclusion" to the conditions to which he would apply it, and in which, as I have said, he tacitly assumes that no relative motion of the parts of the medium takes place. In other cases, also, the inadequacy of his statement may be demonstrated.

There exists a source of thermal instability which does not call for the external constraints of the geyser, and which might be supposed to actuate a geyser of any dimensions whatsoever and which, indeed, in some cases probably takes a part in geyser activity. I refer to the phenomena attending ebullition in liquids which contain no undissolved gases; *i. e.*, no bubbles

It is well known that this condition permits of a rise in temperature of the liquid greatly over the boiling-point. Boiler explosions have been traced to this source as well as unpleasant experiences with wash-bottles which have been left for long standing on the hot-plate. The fact that a sod, or a handful of gravel, cast into the geyser will precipitate eruption supports the view that its explosive character may be—at least in part—traceable to this phenomenon.

Obviously an ideally tranquil ocean, continually receiving a supply of heat from any source, might be brought into a state of instability in this way. The temporary constraint in this case resides in the tensile strength or molecular adhesion of the medium. A tidal movement or some other periodic source of disturbance may then initiate the castastrophic break-down. And we may fancy the rising vapour, condensing into clouds, as being seasonally or periodically restored as rain. Heat is here carried from source to refrigerator intermittently so long as the thermal supply continues. In other words, "successive revolutions" dominate the history of the region.

In the case of the "Air Tester" the conditions are more complex. A net physical analogy arises which I sum in the statement—"There is a steady supply of heat entering the system accompanied by cyclical movements of a working substance with and against gravity."

Interesting possibilities arise out of the "Sublimation Engine," which, although differing from the preceding systems in many respects, leads to the same end—*i. e.*, the cyclic transfer of thermal energy from source to refrigerator. I can—as I write—see no reasons why the principle involved might not be applied to a system of unlimited extent and dimensions. A steady thermal supply from beneath, or originating radioactively in the working substance, causes this to sublime and diffusing upwards to finally condense upon a horizontal ceiling kept cool by a refrigerator. When a considerable deposit has collected, its adhesion gives way and the instability rapidly spreading causes the whole deposit to sink back to its original position. A homely example is the periodic fall of soot from an unswept flue. A source of vibration might be supposed to time the period of break-down. The analogy is obvious and need not be further pursued.

Towards the end of Dr. Jeffreys's comments upon periodic phenomena, we reach what I regard as the most important passage in his paper in so far that for the first time it places a clear and definite issue before us. He writes:—"In the terrestrial case the transference of heat

will be partly by way of latent heat; some liquid will solidify when it rises to the top, and the descending currents will contain suspended solid particles, which will gradually melt as they descend. But this does not affect the argument; the only difference is that in the region where solid is present the gradient is the adiabatic for a mixture of solid and liquid, and not that for a liquid alone. There is no possibility of what Professor Joly calls superheating—that is, rise of temperature far above the melting-point."

The importance of this passage is in the assumption that descending solid particles will gradually melt away as they descend.

Now the accumulation of solid or crystallized magma in the depths is, I believe, as essential to the theory of thermal cycles as it is to Lord Kelvin's contention that a crust forming at the surface of the cooling Earth and breaking up must collect beneath till all is solidified. In fact, the conditions are very much alike, save that in Kelvin's theory the loss of heat is at the very surface. Kelvin was very clear that the gradient of temperature downwards in the cooling Earth must be sufficiently slow as not to annul the effects of pressure in raising the melting-point. Dr. Jeffreys discusses this matter in 'The Earth,' and gives an expression defining the limiting gradient according to Kelvin's "thermo-dynamic law of freezing."

In the theory of thermal cycles this condition is equally essential. The crystals rise in melting-point as they descend. The thermal gradient must not rise steeply enough to counteract this rise of melting-point and cause the remelting of the crystals. This is quite clear, and it directs our attention to extraneous sources of a downward-rising temperature gradient.

Are there deeper-lying sources of heat? Here we must recall the fact that thermal supplies to the surface of the continents, as determined by temperature gradients and conductivity of the rocks, can, mainly or entirely, be ascribed to heat of radioactive origin in the continents themselves; while in what appear to be the deeper-lying basal rocks, we find a diminishing supply of radioactive substances as we go downwards.

We seem, therefore, led to the conclusion that there can be only a very small supply of heat from regions deeper than those with which we are concerned.

Again, we can hardly invoke the contemporaneous output of radioactive heat. For this is too slow: about 3·4 calories per gram in the case of basalt, and 1·7 calories per gram in the case of eclogite, in one million years.

The gradient prevailing in a molten substratum must arise almost entirely from its own previous radioactivity and convective movements. The latter will tend to make the temperature uniform or diminish the temperature gradient. Tidal shearing movements, while operative, will tend towards uniformity of temperature. We cannot evaluate these effects, but it seems plain that the assumption of a sustained rise of temperature downwards adequate to counteract the effects of pressure in raising the melting-point is not allowable unless some sufficient reason can be adduced. Dr. Jeffreys advances no reasons for such a downward rise of temperature —so far as I know.

But if this high gradient does not exist, then Dr. Jeffreys's "steady state" must fail. For the congealed basalt must build up from beneath concurrently with the escape of heat from the higher regions of the magma into the overlying solid ocean-floor; and the original solid state must in this way be restored.

I am reprimanded by Dr. Jeffreys for not citing chapter and verse in support of my denial of his statement (Phil. Mag. May 1926) "that the melting-points of rocks increase with the basicity; so oceanic rocks must on the whole have higher melting-points than continental rocks." My defence is that I thought the experimental data were sufficiently well known to require no citation. I shall now endeavour to make good the omission.

Dr. Jeffreys upholds his statement and gives two references: "Vogt's Econ. Geol. 1926, pp. 207–233; *cf.* also 'Nature,' March 5, 1927, p. 343." The last is a review of Gutenberg's 'Lehrbuch der Geophysik' by Dr. Jeffreys himself, and apparently contains his reference to Vogt. A quotation from the review, although the wording is rather obscure, appears to show that there has been confusion between "melting-point" and "crystallization-point." It is well known that these differ widely. I quote from Dr. Jeffreys's review:—

"There is a remark on p. 52 that the melting point of basaltic rocks at atmospheric pressure is about 200–300° C. lower than that of granite ones, which is given as 1100°. F. W. Clarke gives 1240° for granite, and values from 1060° to 1250° for basalt ('Data of Geochemistry,' 1924, pp. 298–301). J. H. L. Vogt gives 1250° for the crystallization-point of gabbro, which is chemically similar to basalt, and 1000° for granite ('Economic Geology,' 1926, pp. 207–233). The latter estimates refer explicitly to dry material, but it

seems to be generally believed by geologists that in natural conditions the melting-point of granite is more affected by water than that of basic rocks. A reconsideration of the data on this question is overdue ; such a conflict of opinion on some of the most important experimental data of geophysics should not be allowed to persist."

There is nothing in this, nor in the originals referred to, so far as I can find, to sustain Dr. Jeffreys's contention that the melting-point of rocks increases with the basicity ; but quite the contrary. In confusing the melting-point with crystallizing-point serious error is involved.

In the case of "crystallization-point" we deal with solutions : "The relation of the several compounds in a magma is one of mutual solution" (Harker, 'The Natural History of Igneous Rocks,' p. 169). It results "that the order in which the minerals crystallize is not determined by their relative fusibility as separately tested. This *lowering of freezing-point* is the most characteristic property of solutions" (p. 170). The italics are Dr. Harker's.

Again, volatile substances play an important part in determining the temperature of crystallization of the constituents ; and as these substances escape more freely from extruded than from intruded magmas, the former may crystallize at higher temperatures than the latter. Also, volatile substances (such as water) are more abundant in acid than in basic magmas, and hence play a more important part in the crystallization of the former. "It follows that *under like conditions acid magmas crystallize at lower temperatures than basic magmas.*

But there is no doubt as to the fact that the *melting-point* rises with the percentage of silica. Thus Barus and Iddings find as follows :—

	Melting-point.	Silica percentage.
Basalt	1250°	48·49
Hornblende-mica-porphyry	1400°	61·50
Rhyolite	1500°	75·50

These figures bring out strikingly the difference, already referred to, between the melting-point and the crystallization-point. Rhyolite is generally regarded as the volcanic equivalent of a granite. Clarke, who cites these results ('Data of Geochemistry,' 1924, p. 301), continues : "At 1300° the basalt was quite fluid, but at 1700° the rhyolite was still viscid."

Doelter found that basic rocks softened between 1000°

and 1150°, and acidic rocks softened at 1200°–1300° (Phys. Chem. Min. 1905, p. 124).

Clarke (*loc. cit.* p. 297) states the issue clearly. "In the geological interpretation of the melting-points there is one particularly dangerous source of error. We must not assume that the temperature at which a given oxide or silicate melts is the temperature at which a mineral can rystallize from a "magma."

The experiment of floating a piece of granite in molten basalt may easily be made. The conditions are stable, because the crystallization temperature of basalt (1050°) is beneath the melting-point, or rather the decomposition-point, of orthoclase (1170°) and the melting-point of albite (1100°), and, of course, much below the rather indefinite melting-point of quartz. (See J. H. L. Vogt, 'Physical Chemistry of the Magmatic Differentiation of Igneous Rocks,' ii. 1926.)

There remains one more point. Dr. Jeffreys refers to his deduction, on seismic evidence, of a low continental thickness, assuming that this has been verified. A recent paper by Mohorovičič in Gerlands 'Beiträge zur Geophysik' (Bd. xvii. Hept 2) disputes his low estimate, and contends that the thickness of the continental sial is about 40 km. (*loc. cit.* p. 198). The discussion is still proceeding. Meanwhile the following statement by Dr. Jeffreys ('Nature,' Sept. 25, 1926) referring to certain seismic determinations is of interest. He refers to the Kulpa Valley earthquake of 1909, the Wurtemberg earthquake of 1911, the Tauern earthquake of 1923, and the Oppau explosion.

"The observations permit of a rough determination of the depths of the granite and basaltic layers. The former may be about 12 km., the latter about 20 km. Both are subject to an accidental error of about 4 km. In addition there is a possibility of systematic error. Uncertainty as to the depth of focus may allow the thickness of the granite layer to be doubled. . . . *I think, therefore, that determinations of the depths of the layers by means of near earthquakes are not more reliable than those based on the Earth's thermal state, isostatic balance between continents and oceans, and the group velocities of surface-waves,*" *etc.* The italics are mine.

Isostatic considerations appear to favour 37 km. or thereabouts as the continental thickness ('Nature,' Oct. 29, 1927). Thermal estimates based on surface gradients and the radioactivity of the rocks are not very trustworthy.

The point is not very important, and does not directly affect the theory of thermal cycles.

XXII. *Double Excitation of Upper Levels in the Mercury Atoms by Collisions of the Second Kind.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

MAY we make some remarks concerning the very interesting paper of Professor R. W. Wood in the September issue of your Magazine on "Optical Excitation of Mercury, with Controlled Radiating States and Forbidden Lines" (Phil. Mag. [7] iv. p. 466, 1927).

Professor Wood describes the radiation of the mercury resonance lamp which contains some nitrogen. He finds "effects of obscure origin, which can be explained only with difficulty." Very strange is the development of the line 2856 caused by the addition of the nitrogen, whereas the resonance is only excited by light of shorter wave-length than 2750. The line 4916 from the same upper level 3S is also enormously strengthened.

In observations made on the mercury resonance-tube with addition of sodium (*Naturwiss.* xv. p. 540, 1927) we have shown that upper levels of an atom are excited in high selectivity by collisions of the second kind if the energy of the colliding atom can be accepted very completely by the atom struck. At the collision of two metastable (in Wood's terminology $2p_3$) Hg atoms this effect of resonance can have the result that one accepts the whole energy, $2\times 4{\cdot}68$ volts, while the other returns to the normal state. This process has a certain probability, because nearly under the theoretical level of resonance ($2\times 4{\cdot}68=9{\cdot}36$ volts) an actual level of the mercury is located with 9·25 volts, that is, only 0·11 volt is to be transformed into energy of translation; this equals the thermal energy of the two atoms at 300° K. The level from which the mentioned lines 2856 ($2p_2-3S$) and 4916 ($1P-3S$) originate is the level under the point of resonance 3S.

Investigations are under way aiming to substantiate this hypothesis of the accumulation of energy.

Yours faithfully,

H. BEUTLER and B. JOSEPHY.

Kaiser Wilhelm-Institut für physikalisch
Chemie und Elektrochemie, Berlin-Dahlem.
December 6, 1927.

XXIII. *Notices respecting New Books.*

The Nature of the World of Man. By Sixteen Members of the Faculty of the University of Chicago. Edited by H. H. NEWMAN 8vo. Pp. xxiv+566, 132 figs. and 4 charts. (University of Chicago Press; Cambridge University Press, London, 1926. Price 20*s.* net.)

THE purpose of this book is to present an outline of our knowledge of the physical and the biological world, and to show the position of man in the universe in which he lives. It contains the subject-matter of a "survey course" given each year by its authors at the University of Chicago to a group of selected first-year students of superior intelligence. The course was designed to give such students a preliminary view of the rich intellectual fields that lie before them, so that, on the one hand, all of their work shall have a large measure of unity and coherence, and on the other hand, they will be able to decide early what particular subjects they may wish more thoroughly to explore. There has been full co-operation between the various experts engaged in this educational experiment, and its success has seemed to make advisable the production of this volume.

The arrangement adopted is from the general to the specific. Beginning with: I. Astronomy (by Forest Ray Moulton), II. The Origin and Early Stages of the Earth (by Rollin T. Chamberlin) leads naturally to III. Geological Processes and the Earth's History (by J. Harlen Bretz). Physics and Chemistry follow, represented by IV. Energy: Radiation and Atomic Structure (by Harvey Brace Lemon), and V. The Nature of Chemical Processes (by Julius Stieglitz). Biology commences with VI. The Nature and Origin of Life (by Horatio Hackett Newton), followed by VII. a chapter on The Bacteria (by Edwin O. Jordan), after which Merle C. Coulter and Henry Chandler Cowles deal respectively with VIII. Evolution of the Plant Kingdom, and IX. Interaction between Plants and their Environment. General Zoology finds expression in X. The Evolution of the Invertebrates (by W. C. Allee) and XI. The Evolution of the Vertebrates (by Alfred S. Romer), after which XII. The Coming of Man is described by Fay-Cooper Cole. H. H. Newman (the Editor of the series) then discusses XIII. The Factors of Organic Evolution, and Elliot R. Downing XIV. Human Inheritance. XV. Man from the point of view of his Development and Structure (by George W. Bartelmez), XVI. The Dynamics of Living Processes (by Anton J. Carlson), and XVII. Mind in Evolution (by Charles H. Judd), conclude the series. A few selected references are given at the end of each chapter, and a somewhat limited glossary precedes the index.

The volume is well printed and the illustrations, which in part are photographic reproduction in the form of plates and in part simple line text-figures, are clear and helpful.

The book should have a wide appeal to readers who seek a general knowledge of the world about them, and take an interest in the history and future of their own species.

XXIV. *Proceedings of Learned Societies*

GEOLOGICAL SOCIETY.

[Continued from vol. iv. p. 256].

November 2nd, 1927.—Dr. F. A. Bather, M.A., F.R.S., President, in the Chair.

THE following communication was read:—

'The Stratigraphical Distribution of the Cornbrash: I.—The South-Western Area.' By James Archibald Douglas, M.A., D.Sc., Sec.G.S., and William Joscelyn Arkell, B.A., B.Sc., F.G.S.

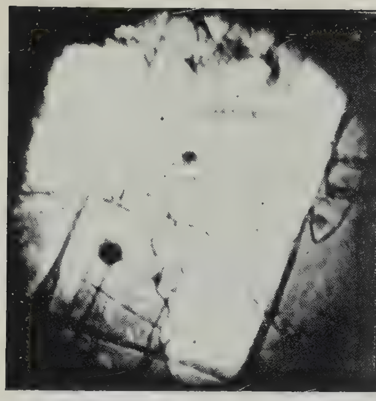
This paper gives an account of the stratigraphical distribution of the Cornbrash in South-Western England, from Oxford to the south coast near Weymouth. The evidence obtained by the Authors is used in a critical discussion of the eleven brachiopod zones proposed by Mr. S. S. Buckman in a recent communication to the Society, and as a result a number of amendments in the list are suggested.

A general description is given of the faunal sequence, and the distribution of the zones throughout the area is indicated by a detailed account of many type-exposures.

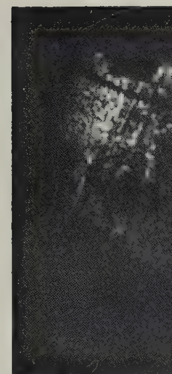
The new records obtained are added to Mr. Buckman's faunal range-diagram, and in this way it is shown that many of his conclusions regarding penecontemporaneous erosion and non-sequences in the Cornbrash, as expressed in his clinal diagram, have been based on insufficient data, and that lack of geological information, in many instances due merely to exposure-failure or faulty collection, has been used as evidence of stratal failure.

The introduction of new stratigraphical terms is criticized, and a twofold rather than a threefold subdivision of the Cornbrash is advocated. The more important brachiopods and lamellibranchs are discussed in palæontological notes, and several new species are described and figured.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]



565° C.



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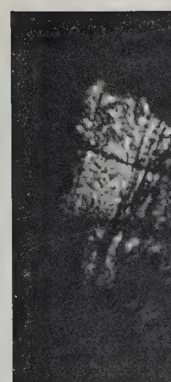
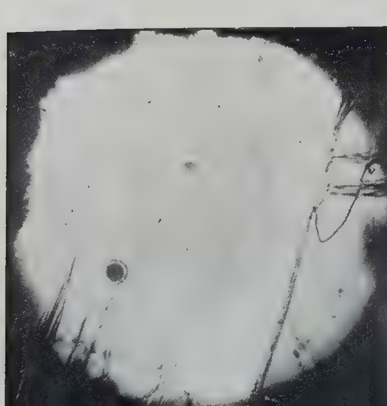
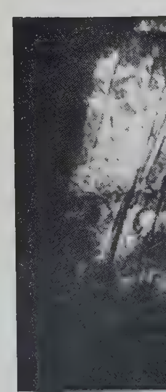
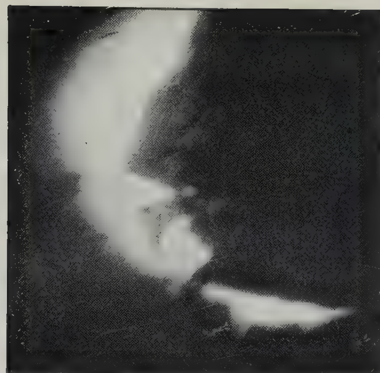




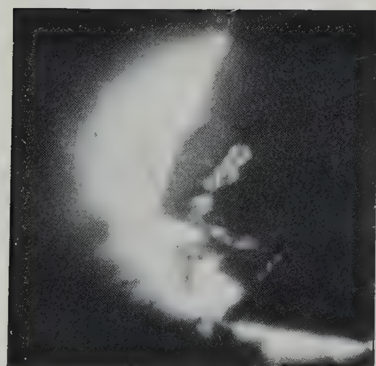
15° C.



Phil. Mag. Ser. 7, Vol. 5, Pl. II.

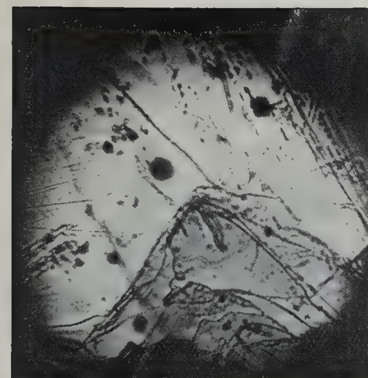


480° C.



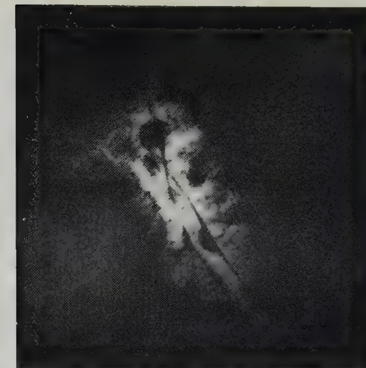
450° C.

15° C.



15° C.

505° C.

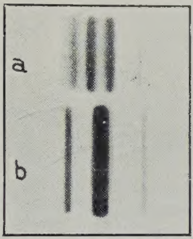


590° C.

515° C.

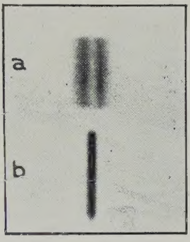
485° C.

FIG. 1.



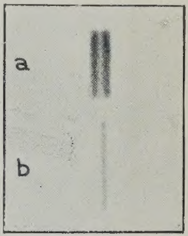
λ 5876

FIG. 2.



λ 6678

FIG. 3.



λ 5016

FIG. 7.

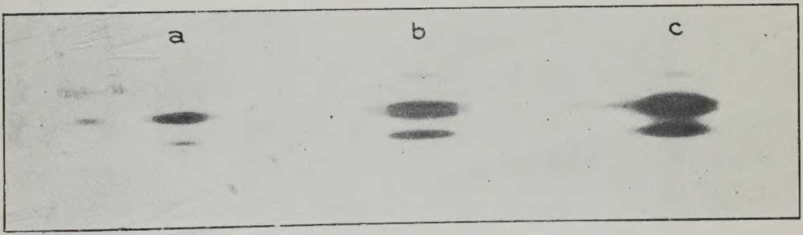


FIG. 8.

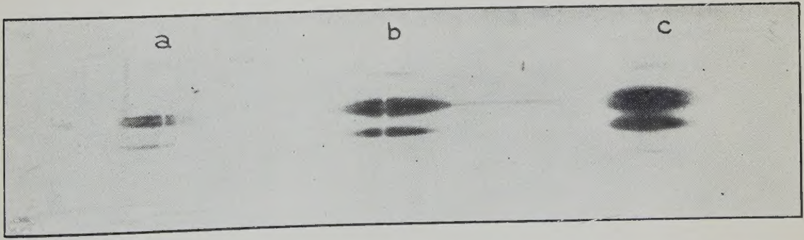
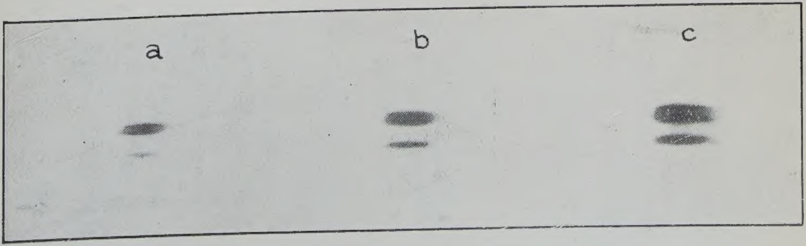
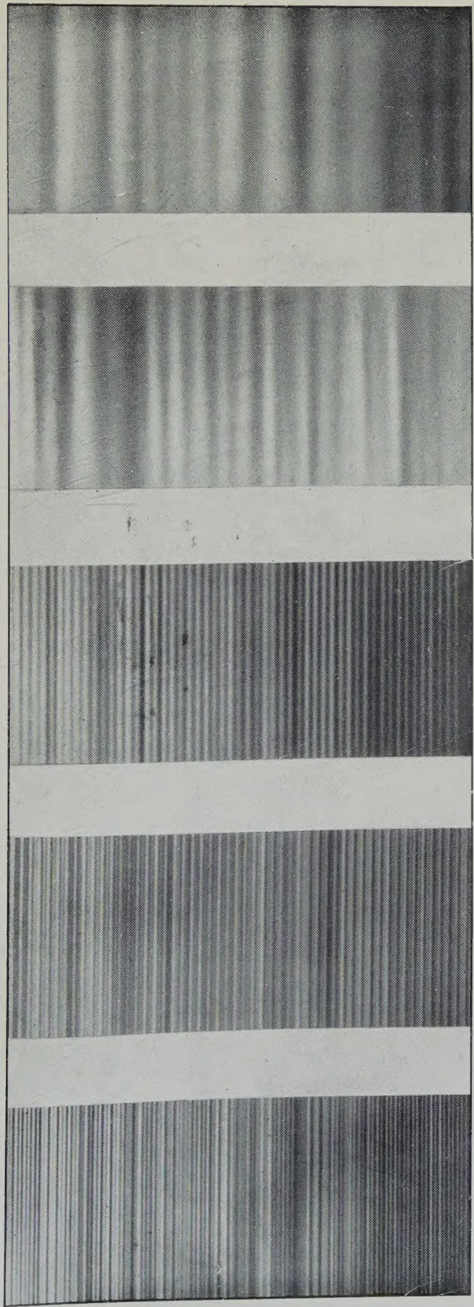


FIG. 9.





A — Principal line alone.

B — Ghost of first order alone.

C — Principal line and ghost of first order.

D — Principal line and ghosts of first and second order.

E — Principal line and four ghosts.

Enlarged about 2 times.